K-POLYSTABILITY OF Q-FANO VARIETIES ADMITTING KÄHLER-EINSTEIN METRICS

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Abstract. It is shown that any, possibly singular, Fano variety X admitting a Kähler-Einstein metric is K-polystable, thus confirming one direction of the Yau-Tian-Donaldson conjecture in the setting of Q-Fano varieties equipped with their anti-canonical polarization. The proof is based on a new formula expressing the Donaldson-Futaki invariants in terms of the slope of the Ding functional along a geodesic ray in the space of all bounded positively curved metrics on the anti-canonical line bundle of X. One consequence is that a toric Fano variety X is K-polystable iff it is K-polystable along toric degenerations iff 0 is the barycenter of the canonical weight polytope P associated to X. The results also extend to the logarithmic setting and in particular to the setting of Kähler-Einstein metrics with edge-cone singularities. Furthermore, applications to bounds on the Ricci potential and Perelman's $\lambda-$entropy functional on $K-$unstable Fano manifolds are given.

Contents

1. Introduction

Let (X,L) be a polarized projective algebraic manifold. i.e. L is an ample line bundle over X. According to the fundamental Yau-Tian-Donaldson conjecture in Kähler geometry (see the recent survey [46]) the first Chern class $c_1(L)$ contains a Kähler metric ω with *constant scalar curvature* if and only if (X,L) is *K-polystable.* This notion of stability is of an algebro-geometric nature and has its origin in Geometric Invariant Theory (GIT). It was introduced by Tian [50] and in its most general form, due to Donaldson [15] it is formulated in terms of polarized $\mathbb{C}^*-$equivariant deformations $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ of (X,L) called *test configurations* for the polarized variety (X,L). Briefly, to any test configuration $(\mathcal{X},\mathcal{L})$ one associates a numerical invariant $DF(\mathcal{X},\mathcal{L})$, called the *Donaldson-Futaki invariant,* and X is said to be K-polystable if $DF(\mathcal{X},\mathcal{L}) \leq 0$ with equality if and only if $(\mathcal{X},\mathcal{L})$ is isomorphic to a product test configuration (the precise definitions are recalled in section 2.2). The test configuration $(\mathcal{X},\mathcal{L})$ plays the role of a one-parameter subgroup

in GIT and the Donaldson-Futaki invariant corresponds to the Hilbert-Mumford weight in GIT. Accordingly, the Yau-Tian-Donaldson conjecture is sometimes also referred to as the manifold version of the celebrated Kobayashi-Hitchin correspondence between Hermitian Yang-Mills metrics and polystable vector bundles.

In the case when the connected component $\mathrm{Aut}(X)_0$ containing the identity of the the automorphism group is trivial, i.e. X admits no non-trivial holomorphic vector fields, it was shown by Stoppa [48] that the existence of a constant scalar curvature metric in $c_1(L)$ indeed implies that (X, L) is K-polystable. The case when $\mathrm{aut}(X)$ is non-trivial leads to highly non-trivial complications, related to the case when $DF = 0$ and was treated by Mabuchi in a series of papers [32, 33]. In this note we will be concerned with the special case when ω is a *Kähler-Einstein metric* of positive scalar curvature. Equivalently this means that the Ricci curvature of ω is positive and constant:

$$\mathrm{Ric}\ \omega = \omega,$$

i.e. L is the anti-canonical line bundle $-K_X$ and X is a Fano manifold. In the seminal paper of Tian [50] it was shown, in the case when $\mathrm{Aut}(X)_0$ is trivial, that X is K-stable along all test configurations $\mathcal{X}$ with normal central fiber $\mathcal{X}_0$ (in particular the central fiber has no multiplicities). Here we will show that the assumption on $\mathrm{Aut}(X)_0$ can be removed, as well as the normality assumption on $\mathcal{X}_0$. In fact, we will allow X to be a general, possibly singular, Fano variety and prove the following

Theorem 1.1. *Let X be a Fano variety admitting a Kähler-Einstein metric. Then X is K-polystable.*

It should be pointed out that, following Li-Xu [30], we assume that the total space $\mathcal{X}$ of the test configuration is normal to exclude some pathological test-configurations that had previously been overlooked in the literature (as explained in [30]). As follows from the results of Ross-Thomas [47] this does not effect the notion of K semi-stability. Moreover, by a remark of Stoppa [49] K-polystability for all normal test configuration is equivalent to having $DF(\mathcal{X}, \mathcal{L}) \leq 0$ for all test configurations with equality iff $(\mathcal{X}, \mathcal{L})$ is isomorphic to a product away from a subvariety of codimension at least two.

We recall that, by definition, X is a Fano variety if it is normal and the anti-canonical divisor $-K_X$ is defined as an ample $\mathbb{Q}-$line bundle (such a variety is also called a $\mathbb{Q}-$Fano variety in the literature) and, following [4], ω is said to be a *Kähler-Einstein metric* on X if ω is a bona fide Kähler-Einstein metric on the regular locus X_{reg} of X and the volume of ω on X_{reg} coincides with the top-intersection number $c_1(-K_X)^n[X]$. The existence of such a metric actually implies that the singularities are rather mild in the sense of the Minimal Model Program in birational geometry [4], more precisely the singularities of X are Kawamata log terminal (klt, for short),

One motivation for considering the general singular setting is that singular Kähler-Einstein varieties naturally appear when taking Gromov-Hausdorff limits of smooth Kähler-Einstein varieties [52, 19]. This is related to the expectation that one may be able to form *compact* moduli spaces of K-polystable Fano varieties if singular ones are included, or more precisely those with klt singularities; compare the discussions in [30] and [39] (where the surface case is considered). From this point of view it may be illuminating to compare the previous theorem with the classical (non-Fano) case of irreducible curves of genus $g \geq 2$. As shown

by Deligne-Mumford, including singular nodal curves X with K_X (=the dualizing sheaf) ample yields a compact moduli space $\bar{\mathcal{M}}_g$ and all such curves are asymptotically Chow and Hilbert stable [35] and in particular K-semistable [47] (see [37] for a recent direct proof of K-stability valid in a higher dimensional setting). The link to the previous theorem comes from the fact that any curve X in $\bar{\mathcal{M}}_g$ admits a Kähler-Einstein metric ω on X_{reg} such that the area of ω on X_{reg} coincides with $c_1(K_X)[X]$. Of course, as opposed to the Fano case the Kähler-Einstein metric on a curve X of genus at least two is *negatively* curved (and complete on X_{reg}). On the other hand, one striking feature of the Fano setting is that it is enough to consider *normal* varieties and even those with "mild singularities" (klt) in the sense of the Minimal Model Program in birational geometry (compare [37, 30]) . Note also that the case of Fano varieties with quotient singularities, i.e. X is a Fano orbifold was previously studied by Ding-Tian [14].

Another motivation for allowing X to be singular comes from the toric setting considered in [3], where it was shown that the existence of a Kähler-Einstein metric on a toric Fano variety X is equivalent to X being $K-$polystable with respect to *toric* test configuration. In turn, this latter property is equivalent to the canonical rational weight polytope P associated to X having zero as its barycenter. However, the question weather the existence of a Kähler-Einstein metric on the toric variety X implies that X is K-polystable for *general* test configurations was left open in [3]. Combining the previous theorem with the results in [3] we thus deduce the following

Corollary 1.2. *A toric Fano variety is K-polystable iff it is K-polystable with respect to toric test configurations. In particular, if P is a reflexive lattice polytope, then the toric Fano variety X_P associated to P is $K-$polystable if and only if* 0 *is the barycenter of P.*

We recall that *reflexive* lattice polytopes P (i.e. those for which the dual P^* is also a lattice polytope) correspond to toric Fano varieties whose singularities are Gorenstein, i.e $-K_X$ is an ample line bundle (and not only a $\mathbb{Q}-$line bundle). This huge class of lattice polytopes plays an important role in string theory, as they give rise to many examples of mirror symmetric Calabi-Yau manifolds [2]. Already in dimension three there are 4319 isomorphism classes of such polytopes [27] and hence including *singular* Fano varieties leads to many new examples of K-polystable and K-unstable Fano threefolds (recall that there are, all in all, only 105 families of *smooth* Fano threefolds).

As explained in section 4 the theorem above extends to the logarithmic setting of Kähler-Einstein metrics on *log Fano variates* (X, D), as considered in [4]. In particular, this shows that if D is an effective $\mathbb{Q}-$divisor with simple normal crossings, and coefficients < 1, on a projective manifold X such that the logarithmic first Chern class of (X, D) contains a Kähler-Einstein metric ω with *edge-cone singularities* along D in the sense of [18, 10, 22], then the pair (X, D) is log K-polystable in the sense of [18, 29, 38]. In a companion paper it will also be shown that any Fano variety admitting canonically balanced metrics, in the sense of Donaldson [17], associated to $(X, -kK_X)$ for k sufficiently large, is K-semistable.

The starting point of the proof of Theorem 1.1 is the following result of independent interest, which expresses the Donaldson-Futaki invariant in terms of the Ding functional $\mathcal{D}$ (see formula 2.8):

Theorem 1.3. *Let X be a Fano variety with klt singularities,* $(\mathcal{X}, \mathcal{L})$ a test configuration for $(X, -K_X)$ (assumed to have normal total space) and ϕ a locally bounded metric on $\mathcal{L}$ with positive curvature current. Setting $t := -\log|\tau|^2$ and denoting by $\phi^t = \rho(\tau)^*\phi_\tau$, the corresponding ray of locally bounded metrics on $-K_X$, the following formula holds:

$$-DF(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi^t) + q,$$

where q is a non-negative rational number determined by the polarized scheme $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ with the property that

- *If $q = 0$ then the central fiber $\mathcal{X}_0$ is generically reduced and $\mathcal{L}$ isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$ on the regular locus of $\mathcal{X}$ (In particular,$\mathcal{X}$ is then $\mathbb{Q}-$Gorenstein)*
- *If $\mathcal{X}_0$ is normal with klt singularities (i.e. the test configuration is special) then $q = 0$.*

In the case when $\mathcal{X}$ is smooth and the support of $\mathcal{X}_0$ has simple crossing we have that $q = 0$ iff $\mathcal{X}_0$ is reduced and $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$.

More precisely, we will give an explicit expression for the number q in the previous theorem, expressed in terms of a given log resolution of the central fiber $\mathcal{X}_0$. The previous theorem should be compared with the results of Paul-Tian [41] and Phong-Ross-Sturm [42, 41] - concerning the general setting of a polarized manifold (X, L) - which express $DF(\mathcal{X}, \mathcal{L})$ in terms of the asymptotic derivative of the Mabuchi functional, plus a correction term taking multiplicities into account, under the assumption that the total space $\mathcal{X}$ be smooth and the metric ϕ on $\mathcal{L}$ be smooth and strictly positively curved. It should also be pointed out that, in the case when $(X, L) = (X, -K_X)$ with X smooth and $\mathcal{X}_0$ normal, Ding-Tian [14] showed that the asymptotic derivative of the Mabuchi functional is equal to the generalized Futaki invariant of $\mathcal{X}_0$.

In order to prove Theorem 1.1 we apply Theorem 1.3 to a weak geodesic ray, emanating from the Kähler-Einstein metric on $-K_X$. We can then exploit a recent result of Berndtsson [7] (and its generalization to singular Fano varieties in [4]) concerning convexity properties of the Ding functional which immediately gives the K-semistability part of Theorem 1.1. As for the proof of Theorem 1.3 it uses, among other things, a result of Phong-Ross-Sturm [42] which in the Fano case expresses the Donalson-Futaki invariant DF in terms of the weight of certain Deligne pairings over the central fiber of the test configuration. We will also use the, closely related, intersection theoretic formulation of the Donaldson-Futaki invariant in [37, 56, 30].

In fact, we will give *two* alternative proofs of Theorem 1.1. The first one is shorter and uses a very recent result of Li-Xu [30]. This latter remarkable result, which confirms a conjecture of Tian, says that it is enough to test K-polystability for *special test configurations* $\mathcal{X}$, i.e. such that the central fiber is normal with klt singularities. This will allow us to restrict our attention to test configurations where the central fiber is a priori known to be reduced, which simplifies the characterization of the the case $DF(\mathcal{X}, \mathcal{L}) = 0$ (which could alternatively also be dealt with using semi-stable reduction as in [30, 1]). The second proof is based on a refined analysis of the singularities of L^2-metrics of certain adjoint direct image bundles, but is in a sense more direct as it uses neither the Minimal Model Program, nor semi-stable reduction.

It may be illuminating to compare the approach here with the original approach of Tian [50] in the case of a non-singular Fano variety. As shown by Tian [50], building on the results in [14]), in the case when $\mathcal{X}$ is a special test configuration, the invariant DF can be computed in terms of the asymptotics of *Mabuchi's K-energy* functional along a one-parameter family ϕ_k^t of smooth metrics induced by a fixed (relatively) projective embedding of $\mathcal{X}$ determined by a sufficiently large tensor power of the relative anti-canonical bundle (which in current terminology is called a *Bergman geodesic* at level k). The sign properties of DF are then determined using that, in the presence of a Kähler-Einstein metric, the Mabuchi's K-energy functional is *proper* (if there are no non-trivial holomorphic vector fields on X), which is the content of deep result of Tian [50]. Here we thus show that the Mabuchi functional and the smooth Bergman geodesic may be replaced by the Ding functional and a weak (bounded) geodesic, respectively, and the properness result with Berndtsson's convexity result. One technical advantage of the Ding functional is that, unlike the Mabuchi functional, it is indeed well-defined along a weak geodesic, as previously exploited in [7, 4] in the context of the uniqueness problem for Kähler-Einstein metrics. Thus the approach in this paper is in line with the programs of Phong-Sturm [43] and Chen-Tang [12] for calculating Donaldson-Futaki invariants by using (weak) geodesic rays associated to test configurations.

In the case when X is a smooth Kähler-Einstein Fano variety with $\mathrm{Aut}(X)_0$ trivial the properness of the Ding functional was shown by Tian [50] as a consequence of his properness result for the Mabuchi functional. It was later shown in [44] that if center of the group $\mathrm{Aut}(X)_0$ is finite then the Ding functional is still proper (in an appropriate sense), but the properness in the case of general Kähler-Einstein manifold is still open. The generalization of the properness result (even when $\mathrm{Aut}(X)_0$ is trivial) to singular Fano varieties and more generally log Fano varieties also appears to be a challenging open problem. Anyway, these subtle issues are bypassed in the present approach.

It should be pointed out that the second point in Theorem 1.3 is not used in the proof of Theorem 1.1. However, as discussed in section 5, it fits naturally into Tian's program [52] for establishing the existence part of the Yau-Tian-Donaldson conjecture - in particular when generalized to the setting of singular Fano varieties. The relation to Tian's program comes from the following immediate consequence of Theorem 1.1 (applied to the universal family $\mathcal{X}$ provided by the universal property of the corresponding Hilbert scheme).

Corollary 1.4. *Let X be a Fano variety embedded in $\mathbb{P}^N$ such that $\mathcal{O}(1)_{|X} = -kK_X$ and ρ the one-parameter subgroup defined by a $\mathbb{C}^*-$action on $\mathbb{P}^N$. Assume that the limiting cycle X_0, as $\tau \to 0$, of the corresponding one-parameter family of varieties $X_\tau := \rho(\tau)_* X$, is normal with klt singularities and that $DF(X_0, \mathcal{O}(1)_{|X_0}) < 0$. Then the the Ding functional (and hence also the Mabuchi functional) tends to infinity along the corresponding curve ϕ_k^t of Bergman metrics on $-K_X$ (i.e. $\phi_k^t := \rho(\tau)^* \phi_{FS|X_\tau}/k$, where ϕ_{FS} is the Fubini-Study metric on $\mathcal{O}(1)$).*

It may be worth stressing that, in Theorem 1.3 and its Corollary above, it is not assumed that the total space $\mathcal{X}$ is smooth and this is why we need to assume that the central fiber $\mathcal{X}_0$ has klt singularities (even if the original Fano variety X is smooth). The point is that this assumption allows us to apply inversion of adjunction [26] to conclude that $q = 0$ in Theorem 1.3 (even though the singularities of $\mathcal{X}$ along $\mathcal{X}_0$ may prevent the central fiber of a log resolution of $(\mathcal{X}, \mathcal{X}_0)$ from being reduced).

More generally, as the proof reveals, the implication that $q = 0$ in Theorem 1.3 holds as long as $\mathcal{X}_0$ is reduced and the log pair $(\mathcal{X}, (1-\delta)\mathcal{X}_0)$ is klt for any sufficiently small positive number δ, which is automatically the case if the total space $\mathcal{X}$ is smooth.

We also give some applications of Theorem 1.3 to bounds on the Ricci potential and Perelman's entropy type $\lambda-$functional [40] (see section 3.3), which can be seen as analogs of Donaldson's lower bound on the Calabi functional [16]. In particular, we obtain the following

Theorem 1.5. *Let X be an $n-$dimensional Fano manifold and set $V := c^1(X)^n$. If X is $K-$unstable, then*

$$\sup_{\omega \in \mathcal{K}(X)} \lambda(\omega) < nV,$$

where $\mathcal{K}(X)$ denotes the space of all Kähler metrics in $c_1(X)$.

As is well-known $\lambda(\omega) \leq nV$ on the space $\mathcal{K}(X)$ and, as recently shown by Tian-Zhang [53] in their study of the Kähler-Ricci flow, if a Fano manifold X admits a Kähler-Einstein metric ω_{KE} then $\lambda(\omega_{KE}) = nV$, or more generally: if Mabuchi's K-energy is bounded from below on $\mathcal{K}(X)$, then supremum of λ is equal to nV. In the light of the Yau-Tian-Donaldson conjecture it seems thus natural to conjecture that X is $K-$semistable if and only if the supremum of λ is equal to nV (the "if direction" is the content of the previous theorem). Finally, it may be worth pointing out that a more precise version of Theorem 1.5 will be obtained, where the supremum of λ is explicitly bounded in terms of minus the supremum of the Donaldson-Futaki invariants over all (normalized) destabilizing test configurations for (X, L) (see Cor 3.10).

Organization of the paper. After having recalled some preliminary material in section 2 the Ding metric associated to a special test configuration $\mathcal{X}$ is introduced and its curvature properties are studied. A proof of Theorem 1.1 can then be given, based on the deep results in [30] concerning special test configurations , which only uses that the Ding metric is positively curved. On the other hand this proof relies heavily on the fact that the central fiber of $\mathcal{X}$ is a priori assumed reduced. In section 3 we introduce the generalized Ding metric associated to any test configuration and study its singularities - in particular how the singularities are related to the multiplicities of the central fiber of the test configuration. The section is concluded with the proof of Theorem 1.3, which then allows us to give an alternative second proof of Theorem 1.1, which is independent of [30]. In section 3.3 the applications to bounds on the Ricci potential and Perelman's entropy functional are given. In section 4 the generalizations to log Fano varieties are explained and in the final Section 5 relations to Tian's program are discussed.

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on the existence problem for Kähler-Einstein metrics on Fano varities. The bound on the functional λ was inspired by a preprint of He [21], which appeared during the revision of the present paper, where different bounds on λ were obtained on Fano manifolds admitting holomorphic vector fields (generalizing [54]).

2. The proof of Theorem 1.1 using special test configurations

2.1. **Setup: Kähler-Einstein metrics on Fano varieties.** Let X be an $n-$dimensional normal compact projective variety. By definition, X is said to be a *Fano variety* if the anti-canonical line bundle $-K_X := \det(TX)$ defined on the regular locus X_{reg} of X extends to an ample $\mathbb{Q}-$line bundle on X, i.e. there exists a positive integer m such that the mth tensor power $-mK_{X_{reg}}$ extends to an ample line bundle over X. Since, X is normal this equivalently means that the anti-canonical divisor $-K_X$ of X defines an ample $\mathbb{Q}-$line bundle. In practice, we will only consider Fano varieties with *klt singularities* (also called *log terminal singularities* in the literature), i.e. there exists a smooth resolution $p: X' \to X$, which is an isomorphism over X_{reg}, such that

$$p^*K_X = K_{X'} + D, \tag{2.1}$$

where $D = \sum c_i E_i$ is a $\mathbb{Q}-$divisor on X' with simple normal crossings for E_i $p-$exceptional with $c_i < 1$ (the analytical characterization of the klt condition will be recalled below). Through out the paper we will use additive notation for line bundles, as well as metrics. This means that a metric $\|\cdot\|$ on a line bundle $L \to X$ is represented by a collection of local functions $\phi(:= \{\phi_U\})$ defined as follows: given a local generator s of L on an open subset $U \subset X$ we define ϕ_U by the relation

$$\|s\|^2 = e^{-\phi_U},$$

where ϕ_U is upper semi-continuous (usc). It will convenient to identify the additive object ϕ with the metric it represents. Of course, ϕ_U depends on s but the curvature current

$$dd^c\phi := \frac{i}{2\pi}\partial\bar{\partial}\phi_U$$

is globally well-defined on X and represents the first Chern class $c_1(L)$, which with our normalizations lies in the integer lattice of $H^2(X, \mathbb{R})$. We will denote by $\mathcal{H}_b(X, L)$ the space of all locally bounded metrics ϕ on L with positive curvature current, i.e. the local representations ϕ_U are all bounded and $dd^c\phi_U \geq 0$ in the sense of currents. Fixing $\phi_0 \in \mathcal{H}_b(X, L)$ and setting $\omega_0 := dd^c\phi_0$ the map $\phi \mapsto v := \phi - \phi_0$ thus gives an isomorphism between the space $\mathcal{H}_b(X, L)$ and the space $PSH(X, \omega_0) \cap L^\infty(X)$ of all bounded ω_0-psh functions, i.e. the space of all bounded usc functions v on X such that $dd^c v + \omega_0 \geq 0$.

2.1.1. *Kähler-Einstein metrics.* In the special case when $L = -K_X$ any given metric on $\phi \in \mathcal{H}_b(X, L)$ induces a measure μ_ϕ on X, which may be defined as follows: if U is a coordinate chart in X_{reg} with local holomorphic coordinates $z_1, ..., z_n$ we let ϕ_U be the representation of ϕ with respect to the local trivialization of $-K_X$ which is dual to $dz_1 \wedge \cdots \wedge dz_n$. Then we define the restriction of μ_ϕ to U as $\mu_\phi = e^{-\phi_U} dz_1 \wedge \cdots \wedge dz_n$. In fact, this expression is readily verified to be independent of the local coordinates z and hence defines a measure μ_ϕ on X_{reg} which we then extend by zero to all of X. The Fano variety X has *klt singularities* precisely when the total mass of μ_ϕ is finite for some and hence any $\phi \in \mathcal{H}_b(X, L)$ (see [4] for the equivalence with the usual algebraic definition involving discrepancies on

smooth resolutions of X). Abusing notation slightly we will use the suggestive notation $e^{-\phi}$ for the measure μ_ϕ. This notation is compatible with the usual notation used in the context of adjoint bundles: if s is a holomorphic section of $L+K_X \to X$ and ϕ is a metric on L then $|s|^2 e^{-\phi}$ may be naturally identified with a measure on X. In particular, letting $L = -K_X$ and taking s to be the canonical section 1 in the trivial line bundle $L + K_X$ gives us back the measure μ_ϕ. More generally, if (X, D) is a log pair (see section 4 below) and ϕ is a locally bounded metric on $-(K_X + D)$ then one obtains a measure μ_ϕ on X by using the natural identification between $-(K_X + D)$ and $-K_X$ on the complement of the support of D in X and extending by zero to all of X (compare [4]). Abusing notation, we will sometimes write $\mu_\phi = e^{-(\phi+\log|s_D|^2)}$, where s_D is the (multi-) section cutting out D. These constructions are compatible with taking resolutions p, as in 2.1: if ϕ is a metric on $-K_X$, then $p^*\phi$ is a metric on $-(K_{X'} + D)$ and $p_*(\mu_{p^*\phi}) = \mu_\phi$.

Following [4] ω is said to be a *Kähler-Einstein metric* on X if it is Kähler metric on X_{reg} with constant Ricci curvature, i.e. Ric $\omega = \omega$on X_{reg} and $\int_{X_{ref}} \omega^n = c_1(-K_X)^n$. As shown in [4] this equivalently means that the Fano variety X in fact has klt singularities and ω extends to a Kähler current defined on the whole Fano variety X, such that ω is the curvature current of a locally bounded (and in fact continuous) metric ϕ_{KE} on the $\mathbb{Q}-$line bundle $-K_X$ such that

$$(dd^c\phi_{KE})^n = Ve^{-\phi_{KE}} / \int_X e^{-\phi_{KE}}. \tag{2.2}$$

The measure appearing the left hand side above is the *Monge-Ampère measure* of ϕ_{KE} defined in sense of pluripotential theory (see [4] and references therein for the singular setting).

2.2. **K-polystability and test configurations.** Let us start by recalling Donaldson's general definition [15] of K-stability of a polarized variety (X, L), generalizing the original definition of Tian [50]. First, a *test configuration* for (X, L) consists of a scheme $\mathcal{X}$ and a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ with a $\mathbb{C}^*-$action ρ on $\mathcal{L}$ and a $\mathbb{C}^*-$equivariant flat surjective morphism $\pi : \mathcal{X} \to \mathbb{C}$ (where the base $\mathbb{C}$ is equipped with its standard $\mathbb{C}^*-$action) such that $(X_1, \mathcal{L}_1)$ is isomorphic to (X, rL) for some integer r. In fact, by allowing $\mathcal{L}$ to be a $\mathbb{Q}-$line bundle we may as well assume that $r = 1$. More generally, for a *semi-test configuration* we only require that $\mathcal{L}$ be relatively semi-ample. The *Donaldson-Futaki invariant* $DF(\mathcal{X}, \mathcal{L})$ of a test configuration is defined as follows: consider the N_k-dimensional space $H^0(X_0, kL_0)$ over the central fiber X_0 and let w_k be the weight of the $\mathbb{C}^*-$action on the complex line $\det H^0(X_0, kL_0)$. Then the Donaldson-Futaki invariant of $DF(\mathcal{X}, \mathcal{L})$ is defined as a the sub-leading coefficient in the expansion of w_k/kN_k in powers of $1/k$ (up to normalization):

$$\frac{w_k(\det H^0(X_0, kL_0))}{kN_k} = c_0 + \frac{1}{k}\frac{1}{2}DF(\mathcal{X}, \mathcal{L}) + O(\frac{1}{k^2}),$$

where $N_k := \dim(H^0(X_0, kL_0)$. The polarized variety (X, L) is said to be *K-semistable* if, for any test configuration, $DF(\mathcal{X}, \mathcal{L}) \leq 0$ and *K-polystable* if moreover equality holds iff $(\mathcal{X}, \mathcal{L})$ is a product test configuration, i.e. $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$. Following [30] we also assume that the total space $\mathcal{X}$ of the test configuration is normal, to exclude some pathological phenomena observed in [30] (then the morphism π is automatically flat; see Prop 9.7 in [23]). We also recall that (X, L)

is said to be $K-$*unstable* if it is not K-semistable, i.e. there exists a *destabilizing test configuration* in the sense that $DF(\mathcal{X},\mathcal{L})>0$.

In this paper we will be concerned with test configurations $(\mathcal{X},\mathcal{L})$ for a Fano variety with its anti-canonical polarization, i.e. X is a Fano variety and $L=-K_X$ so that the restriction of $\mathcal{L}$ to the complement $\mathcal{X}^*$ of the central fiber coincides with the $\mathbb{Q}-$line bundle defined by the dual of the relative canonical divisor $K_{\mathcal{X}/\mathbb{C}}:=K_{\mathcal{X}}-\pi^*K_{\mathbb{C}}$ (which we will sometimes denote by K to simplify the notation). Note that, in general, $K_{\mathcal{X}/\mathbb{C}}$ does not extend as a $\mathbb{Q}-$line bundle over the central fiber, but following [30] we say that a normal variety $\mathcal{X}$ with a $\mathbb{C}^*-$equivariant surjective morphism π to $\mathbb{C}$ is a *special test configuration for the Fano variety* X if $\mathcal{X}_1=X$, the total space $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein and the central fiber is reduced and irreducible and defines a Fano variety with klt singularities. Then we set $\mathcal{L}=-K_{\mathcal{X}/\mathbb{C}}$. Moreover, since the Donaldson-Futaki is independent of the lift of the $\mathbb{C}^*-$action on $\mathcal{X}$ we may and will assume that the $\mathbb{C}^*-$action on $-K_{\mathcal{X}/\mathbb{C}}$ is the canonical lift of the $\mathbb{C}^*-$action on $\mathcal{X}$ to $-K_{\mathcal{X}/\mathbb{C}}$. It will be useful to recall the following essentially well-known characterization of special test configurations:

Lemma 2.1. *Let $(\mathcal{X},\mathcal{L})$ be a general test configuration (with a priori non-normal total space) for $(X,-K_X)$, where X is a Fano variety. Assume that the central fiber $\mathcal{X}_0$ is normal. Then $\mathcal{X}$ and $\mathcal{X}_0$ are both normal $\mathbb{Q}-$Gorenstein varieties and $\mathcal{L}_{|\mathcal{X}_0}$ is isomorphic to $-K_{\mathcal{X}_0}$, i.e. $\mathcal{L}$ is isomorphic to $K_{\mathcal{X}/\mathbb{C}}$. Moreover, if $\mathcal{X}_0$ has klt singularities, then so has $\mathcal{X}$. In other words, a test configuration is special iff the central fiber is reduced with klt singularities.*

Proof. For completeness we provide a proof (thanks to Yuji Odaka for his help in this matter). It follows from general commutative algebra that if $\pi:\mathcal{X}\to\mathbb{C}$ is a morphism projective and flat over $\mathbb{C}$, with normal fibers, then $\mathcal{X}$ is also normal. In particular, the canonical divisor $K_{\mathcal{X}}$ is a well-defined Weil divisor. By assumption $-mK_{\mathcal{X}}$ and $\mathcal{L}$ are Cartier and linearly equivalent on $\mathcal{X}^*$ and hence $-mK_{\mathcal{X}}+\mathcal{L}$ is linearly equivalent to a Weil divisor D supported in the central fiber. But the central fiber is Cartier (since it is cut out by π^*t) and hence, since it is assumed irreducible $-mK_{\mathcal{X}}+\mathcal{L}$ is linearly equivalent to a multiple of $\mathcal{X}_0$, which means that $-mK_{\mathcal{X}}$ is a sum of Cartier divisors, hence Cartier, i.e. $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein. More precisely, $K_{\mathcal{X}}$ is linearly equivalent to $\mathcal{L}$ modulo a pull back from the base and thus it follows from adjunction that the restriction of $\mathcal{L}$ to $\mathcal{X}_0$ is linearly equivalent to $K_{\mathcal{X}_0}$, which concludes the proof of the first statement. Finally, if $\mathcal{X}_0$ has klt singularities it now follows from inversion of adjunction [26] that $\mathcal{X}$ also has klt singularities. □

Very recently Li-Xu [30] used methods from the Minimal Model Program in birational geometry to establish the following result which confirms a conjecture of Tian:

Theorem 2.2. *(Li-Xu)* [30] *Let X be a Fano variety. Then $(X,-K_X)$ is K-polystable iff $DF(\mathcal{X},\mathcal{L})\leq 0$ for any special test configuration for X with equality iff $\mathcal{X}$ is a product test configuration.*

Remark 2.3. We briefly recall that the starting point of the proof in [30] is to replace a given test configuration $\mathcal{X}$ for X by a semi-stable family $\mathcal{Y}\to\mathcal{X}\to\mathbb{C}$. Then the proof proceeds by producing an appropriate special test configuration from $\mathcal{Y}$, by running a relative Minimal Model Program (MMP) with scaling (on the

log canonical modification of $\mathcal{Y}$). The reduction to special test configurations will simplify the proof of Theorem 1.1, but, as explained in section 3.2.1, an independent proof can also be given.

Before continuing we recall [50, 15] that the total space $\mathcal{X}$ of a test configuration may, using the relative linear systems defined by $r\mathcal{L}$ for r sufficiently large, be equivariantly embedded as a subvariety of $\mathbb{P}^N \times \Delta$ so that $r\mathcal{L}$ becomes the pull-back of the relative $\mathcal{O}(1)-$hyperplane line bundle over $\mathbb{P}^N \times \Delta$. We will denote by ϕ_{FS} the metric on $\mathcal{L}$ obtained by restriction of the fiberwise Fubini-Study metrics on $\mathbb{P}^N \times \{\tau\}$.

2.3. **Deligne pairings and the Ding type metric.** The Donaldson-Futaki invariant may also be expressed in terms of Deligne pairings [42]. First recall that if $\pi : \mathcal{X} \to B$ is a proper flat morphism of relative dimension n and $L_0, ..., L_n$ are line bundles over $\mathcal{X}$ then the Deligne pairing $\langle L_0, ..., L_n \rangle$ is a line bundle over B, which depends in a multilinear fashion on L_i [55, 42]. Moreover, given Hermitian metrics $\phi_0, ..., \phi_n$ there is natural Hermitian metric ϕ_D on $\langle L_0, ..., L_n \rangle$ which has the following fundamental properties:

- its curvature is given by

$$dd^c\phi_D = \pi_*(dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n) \tag{2.3}$$

- if ϕ and ψ are metrics in $\mathcal{H}_b(L)$ with ϕ_D and ψ_D denoting the induced metrics on the top Deligne pairing $\langle L, ..., L \rangle$ in the absolute case when B is a point, then we have the following "change of metric formula":

$$\phi_D - \psi_D = (n+1)\mathcal{E}(\phi, \psi) := \int_X (\phi - \psi)(dd^c\phi)^{n-j} \wedge (dd^c\psi)^j \tag{2.4}$$

We also recall (see [4] for the singular setting) that the first variation of the functional $\mathcal{E}(\cdot, \psi)$ on $\mathcal{H}_b(L)$ is given by

$$\frac{d}{dt}_{|t=0}\mathcal{E}(\phi_0(1-t) + \phi_1 t, \psi) = \int (\phi_1 - \phi_0)(dd^c\phi_0)^n \tag{2.5}$$

Let us now come back to the general setting of a test configuration $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ for a polarized variety (X, L). Under appropriate regularity assumptions it was shown in [42] that the following holds:

Proposition 2.4. *(Phong-Ross-Sturm)* [42]*: The Donaldson-Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ is minus the weight over 0 of the following line bundle over $\mathbb{C}$:*

$$\eta := \frac{1}{(n+1)L^n}\left(\mu \langle \mathcal{L}, ..., \mathcal{L} \rangle - (n+1)\left\langle -K_{\mathcal{X}/\mathbb{C}}, \mathcal{L} ..., \mathcal{L} \right\rangle\right),$$

where μ is the numerical constant $n(-K_X) \cdot L^{n-1}/L^n$ expressed in terms of the algebraic intersection numbers on X.

More precisely, it was shown in [42] that, up to natural isomorphisms, the Knudson-Mumford expansion of the determinant line bundle $\det(\pi_*(kL)) \to \Delta$ (with fibers $\det H^0(X_\tau, kL_\tau)$) satisfies

$$\det(\pi_*(kL))/kN_k = \frac{1}{(n+1)L^n}\langle \mathcal{L}, ..., \mathcal{L} \rangle - \frac{1}{k}\frac{1}{2}\eta + O(\frac{1}{k^2})$$

and η is thus naturally isomorphic to the CM-line bundle introduced by Paul-Tian [41]. The proofs in [42] were carried out under the assumption that the total space

$\mathcal{X}$ and the central fiber X_0 be non-singular (in particular, there are no multiple fibers), but as pointed out in [42] the regularity assumptions can be relaxed and in particular the previous proposition applies when $\mathcal{X}$ is a special test configuration for a Fano variety X.

Remark 2.5. For completeness and future reference we note that an alternative direct proof of the previous proposition can be given, which is valid for any $\mathcal{X}$ such that $K_{\mathcal{X}/\mathbb{C}}$ is well-defined as a $\mathbb{Q}-$line bundle (compare Prop 16 [56]). Indeed, as shown in [37, 56, 30], for any normal $\mathcal{X}$ the corresponding DF-invariant may be written as a sum of $n+1-$fold algebraic intersection numbers (we follow the notation in Prop 1 in [30]):

$$(n+1)L^n DF(\mathcal{X},\bar{\mathcal{L}}) := -\mu\bar{\mathcal{L}}\cdot\bar{\mathcal{L}}\cdots\bar{\mathcal{L}} - (n+1)K\cdot\bar{\mathcal{L}}\cdots\bar{\mathcal{L}}, \tag{2.6}$$

computed on the natural equivariant compactification $\bar{\mathcal{X}} \to \mathbb{P}^1$ of $\mathcal{X}$ induced by the projective compactification of the affine line $\mathbb{C}$. Here K denotes the relative canonical (Weil) divisor on the compactified fibration and $\bar{\mathcal{L}}$ denotes extension of $\mathcal{L}$ to the compactification, induced by the fixed action ρ. For a semi-test configuration the latter formula can be taken as the definition of the DF-invariant. In particular, if K is well-defined as a $\mathbb{Q}-$Cartier divisor (i.e. as a $\mathbb{Q}-$line bundle) then it follows from the standard push-forward formula that

$$DF(\mathcal{X},\mathcal{L}) = -\int_{\mathbb{P}^1}\pi_*(\mu(c_1(\bar{\mathcal{L}})^{n+1}) - (n+1)c_1(K)\wedge c_1(\bar{\mathcal{L}})^n),$$

which, according to 2.3 coincides with the degree of the corresponding sum of Deligne pairings over $\mathbb{P}^1$. But, this is nothing but minus the weight over 0 of the $\mathbb{C}^*-$action on η.

We also recall that, as explained in [42], in the case when the total space $\mathcal{X}$ and the central fiber X_0 are smooth and $\mathcal{L}$ is equipped with a smooth metric ϕ with strictly positive curvature along the fibers, there is a naturally induced Mabuchi type metric on η obtained by equipping $K_{X/\mathbb{C}}$ with the metric induced by the volume form $(dd^c\phi)^n$ and taking the metric on η to be the one induced by the sum of the corresponding Deligne pairings. However, in the present paper we will introduce a different metric on η which is defined when X is a Fano variety and $\mathcal{X}$ is a special test configuration (this setting will be generalized to a general test configuration in Section 3).

Thus we assume again that $\mathcal{X}$ is a special test configuration for the Fano variety X so that $\mathcal{L} = -K_{X/\mathbb{C}}$. Then we have that

$$\eta := -\frac{1}{(-K_X)^n(n+1)}\left\langle -K_{\mathcal{X}/\mathbb{C}},...,-K_{\mathcal{X}/\mathbb{C}}\right\rangle,$$

Given a metric ϕ on $-K_{X/\mathbb{C}}$ we will equip η with an induced metric Φ that we will refer to as the *Ding metric:*

$$\Phi := \phi_D + v_\phi, \tag{2.7}$$

where ϕ_D is the usual Deligne metric on η and v_ϕ is the following function on $\mathbb{C}$:

$$v_\phi(\tau) := -\log\int_{X_\tau} e^{-\phi_\tau}$$

We note that ϕ and $\phi + c(t)$ induced the same Ding metric on η. The definition of the Ding metric is made so that, in the absolute case where $\mathbb{C}$ is a replaced with a point and ψ a fixed reference metric on $-K_X$, the functional

$$\mathcal{D}(\phi,\psi) := \Phi - \Psi = -\frac{1}{(-K_X)^n}\mathcal{E}(\phi,\psi) + v_\phi - v_\psi, \tag{2.8}$$

coincides (up to an additive constant) with a *functional* introduced by Ding (see [4] for the singular setting and references therein). As is well-known, this latter functional has the crucial property that its critical points are precisely the Kähler-Einstein metrics. Indeed, by 2.5 and the definition of v_ϕ the critical point equation $d_\phi\mathcal{D}$ =0 is equivalent to the Kähler-Einstein equation

$$(dd^c\phi)^n = e^{-\phi}/\int_X e^{-\phi} \tag{2.9}$$

We will also have use for the following lemma which is an abstract version of the Kempf-Ness approach to the Hilbert-Mumford criterion, used to test stability in Geometric Invariant Theory (compare [46]). Its formulation involves the classical notion of a *Lelong number* $l_\Phi(0)$ at zero of a subharmonic function Φ on the unit-disc in $\mathbb{C}$, which may be defined as the sup over all positive numbers λ such that $\Phi(\tau) \leq \lambda \log|\tau|^2$ close to $\tau = 0$ (equivalently, $l_\Phi(0) = \int_{\{0\}}(dd^c\Phi)$).

Lemma 2.6. *Let F be a line bundle over the unit-disc Δ in $\mathbb{C}$ equipped with a $\mathbb{C}^*-$action ρ compatible with the standard one on Δ and fix an S^1-invariant metric Φ on F with positive curvature current. Then the weight w of the $\mathbb{C}^*$ action on the complex line F_0 may be computed from the following formula:*

$$-\lim_{t\to\infty}\frac{d}{dt}\log\|\rho(\tau)s\|^2_\Phi = w - l_\Phi(0) \tag{2.10}$$

for $t = -\log|\tau|$, s a fixed holomorphic section of F, where $l_\Phi(0)(\geq 0)$.

Proof. When Φ is smooth the lemma is indeed equivalent to the "Hilbert-Mumford criterion" in Geometric Invariant Theory (GIT). In the case when Φ is merely bounded, with positive curvature current, we fix a smooth S^1-invariant metric Φ_0 and write $\Phi = \Phi_0 + U$ where U is a function on Δ. Then $\log\|\rho(\tau)s\|^2_\Phi = \log\|\rho(\tau)s\|^2_{\Phi_0} - U$. Now, for τ close to zero we may write $U(\tau) = \phi(\tau) - \phi_0(\tau)$ for ϕ and ϕ_0 subharmonic functions with ϕ_0 smooth (indeed, $\phi := -\log\|s'\|^2_\Phi$ and $\phi_0 := -\log\|s'\|^2_{\Phi_0}$ for s' a fixed trivializing section of F on a neighborhood of 0). But it is well-known, and easy to verify, that if Φ is an S^1-invariant subharmonic function on a neighborhood of 0 in $\mathbb{C}$ then $-\lim_{t\to\infty}\frac{d}{dt}U(\tau)$ coincides with the Lelong number $l_\phi(0)$ of ϕ at 0. Applying this latter fact to ϕ and ϕ_0 thus concludes the proof. In fact, a direct proof of the lemma can be given along these lines: indeed, we may as well suppose that, in a small neighborhood of $\tau = 0$ there is a trivialization such that $\rho(\tau)s = \tau^w$ and hence minus the derivative in the right hand side of 2.10 is equal to $-\lim_{t\to\infty}\log e^{-tw} - l_\Phi(0)$, as desired. □

It should be pointed out that a variant of the previous lemma was already used (implicitly) in [42] in conjunction with Prop 2.4 to compute the Donaldson-Futaki invariant in terms of asymptotics of Mabuchi's K-energy functional, by equipping η with the Mabuchi type metric. One of the main points of the present paper is the observation that, in the Fano case, the analysis simplifies (and also has the

virtue of extending to the case when X is singular) if one instead equips η with the Ding metric introduced above. This will be explained below, but we first show in the next section how to equip the relative anti-canonical line bundle over $\mathcal{X}$ with a special metric extending a given one on the special fiber X_1. This builds on ideas introduced in the work of Phong-Sturm [43, 45] and Chen-Tang [12].

2.4. **The Monge-Ampère equation on $\mathcal{X}$ and geodesic rays.** Let X be a Fano variety and $\mathcal{X}$ a special test configuration for X. Denote by M the variety with boundary obtained by restricting $\mathcal{X}$ to the unit-disc $\Delta \subset \mathbb{C}$. Given a locally bounded metric ϕ_1 with positive curvature on $-K_X$ we let ϕ be the metric on $-K \to M$ defined as the following envelope:

$$\phi := \sup\{\psi : \ \ \psi \leq \phi_1 \, \text{on} \, \partial M\} \tag{2.11}$$

where ψ ranges over all locally bounded metrics with positive curvature form on $\mathcal{L} \to M$ and ϕ_1 denotes the S^1-invariant metric on ∂M induced by the given metric (since we are not a priori assuming that ψ is continuous the boundary condition above means that, locally, $\limsup_{z_i \to z} \psi(z_i) \leq \phi_1(z)$ for any sequence z_i approaching a boundary point z). Occasionally, we will use the logarithmic real coordinate $t = -\log|\tau|$ on the punctured disc Δ^*. We note that if X is identified with the fiber X_1 of $\mathcal{X}$ then we can use the action ρ to identify the metrics ϕ_τ on X_τ with a curve of metric

$$\phi^t := \rho(\tau)^* \phi_\tau \tag{2.12}$$

on $-K_X$. Next we will show that the metric ϕ above can be seen as a solution to a Dirichlet problem for the Monge-Ampère operator on M. In fact, it will be convenient to formulate the result for any test configuration (where we recall X and the total space $\mathcal{X}$ are always assumed to be normal varieties).

Proposition 2.7. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for the polarized variety (X, L). Then the following holds:*

- *ϕ is S^1-invariant*
- *ϕ is locally bounded with positive curvature current and upper semi-continuous in M*
- *$\phi_\tau \to \phi_1$ uniformly as $|\tau| \to 1$ (with respect to any fixed trivializing of $\mathcal{L}$ close to a given boundary point).*
- *In the interior of M we have that $(dd^c\phi)^{n+1} = 0$ in the sense of pluripotential theory.*

Proof. The first point follows immediately from the extremal defining of ϕ. It will be convenient to identify the metric ϕ_1 with a $\mathbb{C}^*-$invariant metric on $\mathcal{L}$ over the punctured unit-disc Δ^* using the action ρ. We will also, abusing notation slightly, identify the coordinate τ with the psh function on $\pi^*\tau$ on $\mathcal{X}$. Let us first construct a *barrier,* i.e. a continuous metric $\tilde{\phi}$ on $\mathcal{L}$ with positive curvature current such that $\tilde{\phi} = \phi_1$ on ∂M and $\tilde{\phi}_\tau \to \phi_1$ as $|\tau| \to 1$. To this end first observe that for $\epsilon > 0$ sufficiently small there exist a continuous metric ϕ_U with positive curvature on $\mathcal{L} \to U$ over the open set $U := \{|\tau| \leq \epsilon\} \subset \mathcal{X}$. Indeed, we can set $\phi_U = \phi_{FS}$ for the Fubini-Study metric induced by a fixed embedding of $\mathcal{X}$ (see the end of section 2.2). Finally, we set $\tilde{\phi} := \max\{\phi_1 + \log|\tau|, \phi_U - C\}$ for C sufficiently large so that $\tilde{\phi} = \phi_U$ for $|\tau|$ sufficiently small and $\tilde{\phi} = \phi_1 + \log|\tau|$ for $|\tau| > \epsilon/2$. Since $\tilde{\phi}$ is a

candidate for the sup defining ϕ we conclude that

$$\phi \geq \tilde{\phi} \geq \phi_1 + \log|\tau| \tag{2.13}$$

Next, let us show that ϕ is locally bounded from above or equivalently that there exists a constant C' such that

$$\phi \leq \phi_{FS} + C' \tag{2.14}$$

Accepting this for the moment we deduce that the envelope ϕ is finite with positive curvature current. Moreover, the upper bound also implies that the upper semi-continous regularization ϕ^* of ϕ is a candidate for the sup defining ϕ, forcing $\phi = \phi^*$ in the interior of M, i.e. ϕ is upper semi-continuous there. To prove the previous upper bound we note that since any candidate ψ for the sup defining ϕ satisfies $\psi \leq \phi_{FS} + C$ on the set $E := \partial M$ it follows from general compactness properties of positively curved metrics (or more generally, $\omega-$psh functions) that there is a constant C' such that $\psi \leq \phi_{FS} + C'$ on all of M. Indeed, by a simple extension argument we may as well assume that $u := \psi - \phi_{FS}$ extends as an $\omega-$psh function to some compactification $\hat{\mathcal{X}}$ of $\mathcal{X}$ for some semi-positive form current ω with continuous potentials. But since $u \leq C$ on the non-pluripolar set E it then follows from Cor 5.3 in [20] that $u \leq C'$ on all of $\hat{\mathcal{X}}$ (strictly speaking the variety $\hat{\mathcal{X}}$ is assumed non-singular in [20], but we may as well deduce the result by pulling back u to a smooth resolution of $\hat{\mathcal{X}}$). Alternatively, u can be shown to be bounded from above by using the maximum principle to bound it by a solution to a Dirchlet type problem for the Laplace operator wrt a fixed Kähler metric on a resolution of M (compare the argument for the upper bound in [45]).

Let us next consider the behavior of ϕ on Δ^* by identifying ϕ_τ with ϕ^t as above for $t \in [0, \infty[$. Since ϕ is positively curved and S^1-invariant it follows that ϕ^t is convex in t and in particular the right derivative $\dot{\phi}$ wrt t exists at $t = 0$ and $\dot{\phi} \leq (\phi^{t_0} - \phi^0)/t_0$ for any fixed positive number t_0. From the upper bound 2.14 we thus deduce that $\dot{\phi} \leq C_0$, which, combined with the lower bound 2.13, means that there exists a constant C_T such that $\left|\dot{\phi}\right| \leq C_I$ for any $t \in [0, T]$. By convexity this means that $|\phi^t - \phi^0| \leq Ct$ as $t \to 0$ which thus proves the convergence in the third point of the proposition.

As for the final point, the vanishing of the Monge-Ampère measure $(dd^c\phi)^{n+1}$ on the regular part of the interior of M is a standard local argument, which follows from comparison with the solution of the homogenuous Monge-Ampère equation on small balls. Since, the Monge-Ampère measure on a locally bounded metric does not charge pluripolar sets and in particular not the singular locus of M this concludes the proof. $\square$

According to the previous proposition the envelope ϕ thus induces a *weak geodesic ray* ϕ^t (formula 2.12) in the space $\mathcal{H}_b(X, L)$ of all bounded positively curved metrics, starting at a given metric (compare [43]). For much more precise regularity results (given suitably smooth data on ∂M) expressed on a smooth resolution of $\mathcal{X}$ we refer to the paper [45] and to [12]. However, the point here is that the modest regularity results above will be adequate for our purposes and that are valid for any given locally bounded positively curved metric ϕ_1.

2.5. **Curvature properties of the Ding metric associated to a special test configuration.** The key analytical input in the (first) proof of Theorem 1.1 is the

following result which we will deduce from a recent result of Berndtsson [7] and Berndtsson-Paun [8]. It is a variant of results about positivity of direct images previously established in [6] (which concerned smooth metrics). In the case when X is singular we will more precisely be using the generalization obtained in [4] In fact, the more precise result about the vanishing of the Lelong numbers will not be needed fro the (first) proof of Theorem 1.1 - it will be obtained as a special case of the more general situation considered in section 3.

Theorem 2.8. *Let $\mathcal{X}$ be a special test configuration and ϕ an S^1-invariant locally bounded metric on $-K_{\mathcal{X}/\mathbb{C}}$ with positive curvature current. Then the corresponding function $v_\phi(\tau)$ on Δ has the following properties;*

- *$v_\phi(\tau)$ is subharmonic in τ (i.e. convex in t) and its Lelong number at $\tau = 0$ vanishes.*
- *If $v_\phi(\tau)$ is harmonic in τ (i.e. affine in t) then the fibration $\mathcal{X}$ is a product test configuration.*

Proof. Consider the holomorphically trivial restricted fibration $\mathcal{X}^* \to \Delta^*$ over the punctured disc (which is trivialized by action the ρ). In the case when the fibers are smooth Fano varieties it was shown in [7] that $v_\phi(\tau)$ is subharmonic and the result was extended in [4] to the case of Fano varieties with klt singularities. Moreover, since we are assuming that the fibers $\mathcal{X}_\tau$ have klt singularities $v_\phi(\tau)$ is finite for any τ (including $\tau = 0$) and it is thus tempting to conclude that v_ϕ is bounded on Δ (e.g. by appealing to an appropriate continuity result at $\tau \to 0$), but this seems to be non-trivial to prove. Even if the latter boundedness may very well turn out to be true it will be enough for our purposes to know that v_ϕ is bounded from above (which is the only thing needed for the proof of Theorem 2.8) and that the Lelong number of v_ϕ at $\tau = 0$ vanishes (which is used in Cor 1.4) and this is indeed the case for a special test configuration (by Cor 3.3 below).

As for the last point it was shown in [7, 4] that in the case when $v_\phi(\tau)$ is harmonic, for $\tau \in \Delta^*$ (i.e. $v_\phi(e^{-t})$ is affine in t) there is a biholomorphic map F_τ mapping $X_1 \to X_\tau$ such that $F_\tau^*\phi_\tau = \phi_1$ (the results in [7, 4] were formulated in terms of a fixed trivialization, which in our setting means that ϕ_τ is identified with the geodesic ray ϕ^t). Using the equivariant embedding in the end of section 2.2 we can identify F_τ with a family of embeddings of X into $\mathbb{P}^N$. Now, the previous pull-back relation combined with the assumption that ϕ is a locally bounded metric on $-K_{\mathcal{X}/\mathbb{C}} \to \mathcal{X}$ (or equivalently: $\phi - \phi_{FS}$ is a bounded function on the total space of $\mathcal{X} \to \Delta$) we deduce that

$$\sup_X |F_\tau^*\phi_{FS}^{\mathbb{P}^N} - \phi_1| \leq C \tag{2.15}$$

uniformly for $\tau \in \Delta^*$. But then it follows (just as in the proof of Lemma 6.1 in [50]) that F_τ converges, as $\tau \to 0$, to a biholomorphic map between X and X_0, showing that $\mathcal{X}$ is a product test configuration. In fact, as pointed out in [50] this last step only uses that X_0 is reduced. Moreover, since we are assuming that $\mathcal{X}$ is normal it is enough to assume that X_0 is generically reduced. For completeness we recall the argument. First, by 2.15 we may assume that F_τ converges to a holomorphic map F_0 from X_1 onto the complex space $(X_0)_{red}$, underlying the scheme X_0. The map F_0 is finite, since by construction it pull-backs an ample line bundle to another ample line bundle. Moreover, if X_0 is generically reduced then the degree of F_0 is equal to one, i.e. F_0 is generically one-to-one and hence the map $(\tau, x) \mapsto (\tau, F_\tau(x))$

from $X \times \mathbb{C}$ onto $\mathcal{X}$ is a biholomorphism away from a subvariety of codimension two. But then it follows from normality that the map defines a biholomorphism, which concludes the proof. □

Corollary 2.9. *Let $\mathcal{X}$ be a special test configuration and ϕ an S^1-invariant locally bounded metric on $-K_{\mathcal{X}/\mathbb{C}}$ with positive curvature current. Then the corresponding Ding type metric Φ on the top Deligne pairing η of $-K_{\mathcal{X}/\mathbb{C}} \to \Delta$ has the following properties:*

- *Its curvature defines a positive current on Δ.*
- *It is continuous on Δ^*, up to the boundary $\partial\Delta$ and its Lelong number at $\tau = 0$ vanishes.*
- *If the curvature vanishes on Δ^* then $\mathcal{X}$ is a product test configuration.*

Positivity: first we note that the curvature of the usual (non-twisted) Deligne metric ϕ_D on η is non-negative, as follows immediately from the push-forward formula 2.3 (and a standard approximation argument). Equivalently, since ϕ_D is bounded from above (see below) it is enough to consider the holomorphically trivial case over Δ^* where the result amounts to a well-known property of the functional $\mathcal{E}$ (see [4]). Combined with the positivity in the previous theorem this shows that Φ has positive curvature current.

Continuity etc: Let us first verify that if ϕ is a locally bounded positively curved metric on $\mathcal{L}$ over the special test configuration $\mathcal{X}$ then the Deligne metric ϕ_D on η is locally bounded on Δ and continuous at the boundary of Δ.To this end, we first recall that if ψ is a smooth (in a suitable sense) metric on $\mathcal{L}$ then it was shown by Moriwaki [34] that the corresponding Deligne metric ψ_D on the top Deligne product of $\mathcal{L}$ is continuous. But since ϕ is a locally bounded metric on $\mathcal{L}$ we have that $u := \phi - \psi$ is a bounded function on $\mathcal{X}$ and hence it follows from the change of metric formula 2.4 that

$$|\psi_D - \phi_D| \leq (\sup_{\mathcal{X}} |u|) c^1(L)^n$$

is bounded (where L denotes the restriction of $\mathcal{L}$ to a generic fiber) and hence ϕ_D is locally bounded, as desired. Alternatively, the local boundedness of ϕ_D can be verified directly by induction over the relative dimension, using the recursive definition of ϕ_D [55]. Similarly, the continuity at $\tau = 1$ follows from continuity properties at $\tau = 1$ of ϕ_τ. Indeed, by Prop 2.7 we have that $\phi_\tau \to \phi_1$ uniformly as $\tau \to 1$, i.e. $\phi^t \to \phi^0$ and hence it follows from the change of metrics formula that, in a fixed local trivialization close to $\tau = 1$, we have

$$|(\rho(\tau)\phi_\tau)_D - (\phi_1)_D| \leq C' \sup_{X_\tau} |(\rho(\tau)^*\phi_\tau) - \phi_1| \to 0$$

and also $v_\phi(\tau) \to v_\phi(1)$. This shows in particular that, over Δ^*, the Ding metric $\Phi(= \phi_D + v_\phi)$ may be identified with a subharmonic locally bounded S^1-invariant function which is continuous up to $\partial\Delta$. But then it is convex as a function of t and hence continuous on $\Delta^* - \{0\}$. Moreover, since ϕ_D is bounded in a neighborhood of $\tau = 0$ the Lelong number of Φ at $\tau = 0$ is equal to the Lelong number of v_ϕ, which vanishes according to the previous theorem.

Finally, if the curvature vanishes on Δ^* then it follows from the previous theorem that $v_\phi(e^{-t})$ is affine and hence the fibration $\mathcal{X}$ is a product test configuration.

2.6. **Completion of the proof of Theorem 1.1 using special test configurations.** Let ϕ_1 be a fixed metric in $\mathcal{H}_b(-K_X)$ and denote by ϕ the corresponding envelope 2.11. Since the corresponding Ding type metric on η is positively curved and locally bounded on Δ^* with vanishing Lelong number at $\tau = 0$ (by Cor 2.9) it follows from Prop 2.4 combined with Lemma 2.6 and the definition of the Ding functional (formula 2.8) that the Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L})$ of a test configuration, which by Theorem 2.2 may be assumed to be special, may be computed as

$$-DF(\mathcal{X}, \mathcal{L}) := \lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\rho(\tau)^*\phi_\tau, \phi_1) =: \lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(t)$$

Let us now take ϕ_1 to be a Kähler-Einstein metric so that $\frac{d}{dt}\mathcal{D}(t) \geq 0$ for $t = 0$. Of course, if $\phi^t := \rho(\tau)^*\phi_\tau$ were smooth in t, then equality would hold. Anyway, the inequality is all we need (and it follows easily from the continuity of ϕ^t as $t \to 0$, combined with the convexity of $\mathcal{E}$ wrt the affine structure and the dominated convergence theorem applied to v_ϕ). Now by Cor 2.9 $\mathcal{D}(t)$ is convex and hence $\lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(t) \geq \frac{d}{dt}_{t=0^+}\mathcal{D}(t) \geq 0$, i.e. $DF(\mathcal{X}) \leq 0$. Moreover, by convexity $DF(\mathcal{X}) = 0$ iff $\mathcal{D}(t)$ is affine and hence it follows from Cor 2.9 that $\mathcal{X}$ is a product test configuration which thus concludes the proof.

3. Singularity structure of the generalized Ding metric

In this section we will prove a general formula expressing the Donaldson-Futaki invariant of a general test configuration in terms of the Ding functional and a correction term (which will prove Theorem 1.3 in the introduction and provide a second proof of Theorem 1.1, not relying on [30]). To this end we are led to introduce a generalized Ding metric.

Let $(\mathcal{X}, \mathcal{L})$ be test configuration for a Fano variety $(X, -K_X)$ and fix an equivariant log resolution $p : \mathcal{X}' \to \mathcal{X}$ of $(\mathcal{X}, \mathcal{X}_0)$ and write $\mathcal{L}' := p^*\mathcal{L}$. Then $(\mathcal{X}', \mathcal{L}')$ is a semi-test configuration for $(X, -K_X)$. In order to define a generalized Ding metric we first assume, to fix ideas, that the original Fano variety X is smooth with $\mathcal{L}$ a line bundle over $\mathcal{X}$ and define a new line bundle $\delta' \to \mathbb{C}$ by

$$\delta' := -\frac{1}{L^n(n+1)} \langle \mathcal{L}', ..., \mathcal{L}' \rangle + \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C}, \tag{3.1}$$

(when X is smooth the direct image sheaf $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})$ is indeed a line bundle, as explained below). Given a metric ϕ on $\mathcal{L} \to \mathcal{X}$ we denote by Φ' the *generalized Ding metric* on δ', defined as the Deligne metric on the Deligne pairing twisted by the L^2-metric on $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})$, induced by $\phi' :== p^*\phi$. Note that in general $\mathcal{L}$ is only assumed to be a $\mathbb{Q}-$line bundle, i.e. $r\mathcal{L}$ is a line bundle for some positive integer r and then we may simply define $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) := \pi'_*(r(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})/r$ as a $\mathbb{Q}-$line bundle (which is easily seen to be independent of r) and let Φ' be the metric defined by the corresponding $L^{2/r}-$norm (compare the general setting in [9]).

Turning to the case of a general Fano variety X with klt singularities, first recall that, since the variety $\mathcal{X}^*(:= \mathcal{X} - \mathcal{X}_0)$ is klt we have $p^*K_\mathcal{X} = K_{\mathcal{X}'} + D^*$ on $\mathcal{X}' - \mathcal{X}'_0$ for a (sub) klt $\mathbb{Q}-$divisor D^*, whose closure in $\mathcal{X}'$ we will denote by D. We can uniquely decompose $D = D' - E'$ as a sum of effective divisors. We may and will assume that the log resolution is such that the support of D has simple normal

crossings and is transversal to $\mathcal{X}'_0$ (even if this assumption will not be used). We then define

$$\delta' := -\frac{1}{L^n(n+1)}\left\langle \mathcal{L}', ..., \mathcal{L}' \right\rangle + \pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C},$$

and denote by Φ' the corresponding metric on δ', which is defined using the log adjoint L^2-metric on $\pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}})$ (defined wrt a fixed multi-section $s_{D'}$ cutting out D'). To see that $\pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}})$ is indeed a line bundle over $\mathbb{C}$ first note that over $\mathbb{C}^*$, where the sheaf is globally free, any fiber may be identified with $H^0(X', E')$, where $X' = p^*X$ and E' is $p-$exceptional, so that $\dim H^0(X', E') = 1$. The extension property to all of $\mathbb{C}$ then follows from general principles: the direct image sheaf is clearly torsion-free and since the base is a curve any torsion-free sheaf is automatically locally free (see [23]).

3.1. **The Lelong number of the generalized Ding metric.** In this section we will study the Lelong number l_0 of the generalized Ding metric at $\tau = 0$ (as explained in section 2.5 the metric is continuous away from $\tau = 0$). The key result is the following proposition, which complements the general results of Berndtsson-Paun [8], which imply that $l_0 \geq 0$.

Proposition 3.1. *Assume that $\mathcal{X}$ is a smooth variety and $\pi : \mathcal{X} \to \mathbb{C}$ a proper projective morphism over $\mathbb{C}$ which is non-singular (i.e. a submersion) over the punctured disc Δ^* and such that the support of the central fiber $\mathcal{X}_0$ has simple normal crossings. Let $\mathcal{L} \to \mathcal{X}$ be a semi-positive line coinciding with $-K_{\mathcal{X}/\mathbb{C}}$ over Δ^* such that $H^0(\mathcal{X}, \mathcal{L} + K_{\mathcal{X}})_{|\mathcal{X}_0}$ is non-trivial and ϕ a (possible singular) metric on $\mathcal{L}$ with positive curvature current and such that, under restriction, $e^{-\phi}$ is locally integrable on each component of $\mathcal{X}_0$.*

- *If $\mathcal{X}_0$ is reduced, then the Lelong number l_0 at $\tau = 0$ of the induced L^2-metric on the line bundle $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ vanishes.*
- *If ϕ is locally bounded, but $\mathcal{X}_0$ is possibly non-reduced then*

$$l_0 = \max_i \frac{m_i - 1 - c_i}{m_i}$$

where c_i is the order of vanishing along the reduced component E_i of $\mathcal{X}_0$ of a trivializing (multi-) section s of $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ and m_i is the order of vanishing of $\pi^*\tau$.

Proof. For simplicity we first assume that $\mathcal{L}$ is a line bundle (and not only a $\mathbb{Q}-$line bundle). Fix a local trivializing section of $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ over a neighborhood V of $0 \in \mathbb{C}$. It may be identified with a global holomorphic section s of $\mathcal{L} + K_{\mathcal{X}/\mathbb{C}} \to \mathcal{X}_{|V}$. Fix a local coordinate τ on $\mathbb{C}$ and let $v(\tau) := -\log \|s\|^2_{L^2_\Phi}$ be the corresponding local weight of the L^2-metric on $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$, which is subharmonic functions on V (by [8, 9]). It will be useful to write the Lelong number l_0 of v at 0 as follows

$$l_0 = \inf\left\{ l : \int_V e^{-(v(\tau)+(1-l)\log|\tau|^2)} d\tau \wedge d\bar{\tau} < \infty \right\} \tag{3.2}$$

Noting that $|s|^2 e^{-\phi} \otimes d\tau \wedge d\bar{\tau}$ defines measure on $\mathcal{X}_{|V}$ we can rewrite the integral above as

$$\int_V e^{-(v(\tau)+(1-l)\log|\tau|^2} d\tau \wedge d\bar{\tau} = \int_{\mathcal{X}_{|V}} |s|^2 e^{-\phi} e^{-(1-l)\log|\tau|^2} \otimes d\tau \wedge d\bar{\tau}$$

Now assume that $\mathcal{X}_0$ is reduced and fix a point $x_0 \in \mathcal{X}_0$ and a small neighborhood U of x_0. We may assume that $\mathcal{L}$ and $\mathcal{X}$ are holomorphically trivial over U and that τ can be complemented with a local coordinate $z \in \mathbb{C}^n$ such that (τ, z) define local holomorphic coordinates on U. In the corresponding trivialization of $\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}$ we may write the measure $|s|^2 e^{-\phi} \otimes d\tau \wedge d\bar{\tau}$ as $|s|^2 e^{-\phi} dz \wedge d\bar{z} \wedge d\tau \wedge d\bar{\tau}$ where s is identified with a local holomorphic function and ϕ with a local psh function. Since $\mathcal{X}_0$ is assumed to be reduced we have that τ vanishes to order one along any reduced component E_i. If now $p \in E_i$ it thus follows from the Ohsawa-Takegoshi theorem (see section 2 in [13] for the precise formulation needed here) that

$$\int_U e^{-\phi} e^{-(1-l)\log|\tau|^2} \otimes d\tau \wedge d\bar{\tau} \leq C \int_{U \cap \{s_i = 0\}} e^{-\phi} dV_{n-1} < \infty \tag{3.3}$$

where the finiteness follows from the very assumption on ϕ. Since the point x_0 was arbitrary an l is any given constant in $]0,1[$ this shows that $l_0 = 0$ in formula 3.2, as desired.

Turning to the proof of the second point we assume that ϕ is locally bounded, but that $\mathcal{X}_0$ is possibly non-reduced. Given a point $x_0 \in \mathcal{X}_0$ we may assume that $z = (z_1, ..., z_n)$ and $\tau = z_1^{m_1} \cdots z_r^{m_r}$ so that

$$\int_U |s|^2 e^{-\phi} e^{-(1-l)\log|\tau|^2} \otimes d\tau \wedge d\bar{\tau} \sim \int \prod_{i=1,..r} \frac{1}{|z_i^2|^{(1-l)m_i - c_i}} i dz_i \wedge d\bar{z}_i$$

Since the integrability exponent of $\frac{1}{|z_i^2|}$ is equal to one this concludes the proof of the second point. The case when $r\mathcal{L}$ is a line bundle is treated in exactly the same way, after replacing $|s|^2$ with $|s|^{2/r}$ and ϕ with ϕ/r.

For completeness we also give a proof of the positivity of the curvature of the $L^{2/r}-$metric [8], which by basic extension properties of subharmonic functions amounts to showing that $v(\tau) \leq C$ on V. First, in the case when $r = 1$ it follows from the Ohsawa-Takegoshi extension theorem there exists a constant A independent of $\tau \neq 0$ such that, for any holomorphic section $s^{(\tau)} \in H^0(\mathcal{X}_\tau, L_{|\mathcal{X}_\tau} + K_{\mathcal{X}_\tau})$ there exists $S^{(\tau)} \in H^0(\mathcal{X}_{|V}, \mathcal{L} + K_{\mathcal{X}^*/\mathbb{C}})$ such $S^{(\tau)}_{|\mathcal{X}_\tau} = s^{(\tau)} \otimes d\tau$ and $\left\| S^{(\tau)} \right\|_{\mathcal{X}_{|V}} \leq A \left\| s^{(\tau)} \right\|_{\mathcal{X}_\tau}$ (in terms of the L^2-norms induced by ϕ). In our case $H^0(\mathcal{X}_\tau, L_{|\mathcal{X}_\tau} + K_{\mathcal{X}_\tau})$ is one-dimensional and since $H^0(\mathcal{X}, \mathcal{L} + K_{\mathcal{X}})_{|\mathcal{X}_0}$ is assumed non-trivial $S := S^{(\tau)}$ can thus be taken to be independent of τ and not identically zero giving $\int_{\mathcal{X}_{|V}} |S|^2 e^{-\phi} e^{-(1-l)\log|\tau|^2} \otimes d\tau \wedge d\bar{\tau} \leq A e^{-v(\tau)}$. But since ϕ is locally bounded from above and S is non-trivial the lhs in the previous inequality is trivially bounded from below by a positive constant (just integrate over a fixed small ball in $\mathcal{X}_{|V}$ where $S \neq 0$) and hence $v(\tau) \leq C$ as desired. For a general r the same argument applies if one instead uses the $L^{2/r}-$version of the Ohsawa-Takegoshi extension theorem established in [9]. □

Remark 3.2. Since the induced metric on $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ is positively curved we have $l_0 \geq 0$ and hence $c_{i_0} \leq m_{i_0} - 1$, where i_0 is the index realizing the maximum in the second point of the previous proposition. This is in general false if the assumption that $H^0(\mathcal{X}, \mathcal{L} + K_{\mathcal{X}})_{|\mathcal{X}_0}$ be non-trivial is removed.

Corollary 3.3. *Let $(\mathcal{X}, \mathcal{L})$ be a special test configuration for a Fano variety $(X, -K_X)$ with klt singularities and ϕ locally bounded metric on $\mathcal{L}$ with positive curvature current. Then the Ding metric has vanishing Lelong number at $\tau = 0$.*

Proof. Strictly speaking this is not a corollary of the previous proposition. But if, for example, we knew that $\mathcal{X}'_0$ were reduced for some resolution then the corollary would follow immediately from the previous proposition applied to the resolution $\mathcal{X}'$ (also using the klt condition to get a klt divisor D' on $\mathcal{X}'$). We will instead apply the general algebraic form of inversion of adjunction directly on $\mathcal{X}$. By assumption $\mathcal{X}_0(:= (\pi^*\tau = 0))$ defines a reduced Cartier divisor in $\mathcal{X}$ such that the underlying variety has klt singularities. But then it follows from Theorem 7.5 and Corollary 7.6 in [26] that $(\mathcal{X}, (1-\delta)\mathcal{X}_0))$ is a klt pair in a neighborhood of $\mathcal{X}_0$, which implies the integrability property 3.3 (for $l = \delta$) when reformulated in analytic terms (using that ϕ is assumed locally bounded); see for example [4]. Note that this argument does not imply that, on a resolution, $\mathcal{X}'_0$ is reduced since there may be cancellations coming from the coefficients c_i. $\square$

3.2. **Expressing the Donaldson-Futaki invariant in terms of the Ding functional for a general test-configuration.** It follows immediately from the projection formula applied to the the intersection theoretic formula 2.6 for $DF(\mathcal{X}, \mathcal{L})$ that, setting

$$\eta' := -\frac{1}{(n+1)L^n}\left\langle \mathcal{L}', ..., \mathcal{L}' \right\rangle + \frac{1}{L^n}\left\langle \mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} + D', \mathcal{L}'..., \mathcal{L}' \right\rangle,$$

where η' is thus a line bundle over $\mathbb{C}$ defined in terms of Deligne pairings on the fixed log resolution, we have

$$DF(\mathcal{X}, \mathcal{L}) = DF(\mathcal{X}', \mathcal{L}') = w_0(\eta')$$

Lemma 3.4. *We have that* $-DF(\mathcal{X}, \mathcal{L}) \geq w_0(\delta')$

Proof. Using $w_0(\eta) = w_0(\eta')$ and decomposing

$$\eta' = \delta' + \left(\frac{1}{L^n}\left\langle K_{\mathcal{X}'/\mathbb{C}} + D' + \mathcal{L}', \mathcal{L}', ..., \mathcal{L}' \right\rangle - \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} + D') \right) \tag{3.4}$$

reveals that it is enough to show that $w_0\left(\frac{1}{L^n}\left\langle K_{\mathcal{X}'/\mathbb{C}} + D' + \mathcal{L}', \mathcal{L}', ..., \mathcal{L}' \right\rangle - \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) + D'\right) =$

$$= (K_{\bar{\mathcal{X}}'/\mathbb{P}^1} + D' + \bar{\mathcal{L}}') \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' - \deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1} + D') \geq 0,$$

where we have used the the compactification $\bar{\mathcal{X}}'$ of the resolution $\mathcal{X}'$ and the corresponding extension $\bar{\mathcal{L}}'$ of $\mathcal{L}'$ in the first equality (compare Remark 2.5). To simplify the notation we consider the case when X is smooth so that $D' = 0$, but the general case is essentially the same. Note that the formula above involving the degrees is invariant under $\mathcal{L}' \to \mathcal{L}' \otimes \pi'^*\mathcal{O}_{\mathbb{P}^1}(m)$ and hence we may as well assume that $\deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) = 0$ (this corresponds to a performing an overall twisting of the original action ρ on $\mathcal{L}$). But the latter vanishing means that the line bundle $\pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) \to \mathbb{P}^1$ admits a global non-trivial holomorphic section s, unique up to scaling by a non-zero complex constant. In particular, s induces a global holomorphic section $\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1} \to \mathbb{P}^1$. This means that $\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}$ is linearly equivalent to an effective divisor E (whose support is contained in the central fiber). But then it follows, since $\bar{\mathcal{L}}'$ is relatively semi-ample, that

$$(K_{\bar{\mathcal{X}}'/\mathbb{P}^1} + \bar{\mathcal{L}}') \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' = E \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \geq 0 \tag{3.5}$$

which thus concludes the proof. $\square$

Now we can prove the following more precise version of Theorem 1.3, stated in the introduction:

Theorem 3.5. *Let X be a Fano variety with klt singularities and $(\mathcal{X}, \mathcal{L})$ a test* configuration (with normal total space) for $(X, -K_X)$ with ϕ denoting a locally bounded metric on $\mathcal{L}$ with positive curvature current. Then, setting $\phi^t := \rho(\tau)^*\phi_\tau$, we have

$$-DF(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi^t) + q, \tag{3.6}$$

where q is a non-negative rational number determined by the polarized central fiber $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ with the following properties

- *If $q = 0$, then $\mathcal{X}_0$ is generically reduced and $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein with $\mathcal{L}$ isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$.*
- *If the central fiber of $\mathcal{X}$ is a normal variety with klt singularities (equivalently: the test configuration is special) then $q = 0$.*
- *More precisely, if $(\mathcal{X}', \mathcal{X}_0')$ is a given log resolution of $(\mathcal{X}, \mathcal{X}_0)$ with E_i denoting the reduced components of $\mathcal{X}_0'$, then the following formula holds*

$$q = \max_i \frac{m_i - c_i - 1}{m_i} + \frac{1}{L^n}\sum_i c_i \mathcal{L}'^n \cdot E_i, \tag{3.7}$$

  where m_i and c_i are the order of vanishing along E_i of $\mathcal{X}_0'$ of $\pi'^*\tau$ and any given non-trivial meromorphic (multi-) section s' of $\mathcal{L}' + \mathcal{K}_{\mathcal{X}'/\mathbb{C}} + D' \to \mathcal{X}'$, respectively.
- *In the case when $\mathcal{X}$ is smooth and the support of $\mathcal{X}_0$ has simple normal crossing we have that $q = 0$ iff $\mathcal{X}_0$ is reduced and $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$.*

Proof. Let us start by proving the formula 3.6 in the third point. To simplify the notation we consider the case when X is smooth so that $D' = 0$, but the proof in the general case is essentially the same.

Fix a trivializing section s' of $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C}$. The section s' induces an isomorphism between $\mathcal{L}$ and $-K_{\mathcal{X}^*/\mathbb{C}^*}$ over $\mathcal{X}^*$. In fact, since the formula for $DF(\mathcal{X}, \mathcal{L})$ is invariant under an overall twist of the action on $\mathcal{L}$ we may as well assume that s' is an invariant section and hence, using the notation in the previous lemma $\deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) = 0$. We also fix a trivializing section s_0 of the top Deligne pairing of $\mathcal{L}$ over $\tau = 0$. By Lemma 2.6

$$w(\delta') = -\lim_{t\to\infty} \frac{d}{dt} \log \|\rho(\tau)S_0\|^2_{\Phi'} + l_0,$$

where $S_0 = s_0 \otimes s_0' \in \delta'_{|\tau=0}$ and $l_0 (\geq 0)$ is the Lelong number of the metric Φ' on δ'. Moreover, setting $\phi^t = \rho(\tau)^*\phi_\tau$ we can write

$$-\log \|\rho(\tau)S_0\|^2_{\Phi} = -\log \|s_0\|^2_{\phi^t} - \log \|s_0'\|^2_{\phi^t} = -\frac{1}{L^n}\mathcal{E}(\phi_t) + 0 + \log \int_X e^{-\phi_t} + c_0$$

using the previous identifications, where c_0 is a fixed constant which comes from the change of metrics formula for the Deligne pairing 2.4. Finally, using $-DF(\mathcal{X}, \mathcal{L}) = w_0(\eta')$ and the decomposition formula in Lemma 3.4 together with formula 3.5 this proves formula 3.6. Note that if we would have chosen another trivializing section $\tilde{s}$ then $\tilde{s} = e^{f(\tau)}s$ where $f(\tau)$ is harmonic function on Δ, which corresponds to changing ϕ^t to $\phi^t + f(e^{-t})$ which anyway leaves the Ding function invariant and the Lelong number invariant, as it must. As for the formula 3.7 it follows from Prop 3.1. More precisely, when s' defines a trivialization of the corresponding direct image bundle the formula follows immediately from the previous proposition. Now, a general section may be written as $f(\tau)s'$ for $f(\tau)$ a meromorphic function,

whose vanishing (or pole) order at $\tau = 0$ we denote by m. Since the formula for q is invariant under $c_i \to c_i + m$ the case of a general section thus follows.

To prove the first point we assume that $q = 0$ (this is the direction that is used in the proof of Theorem 1.1). We take s' to be defined by a trivialization section as above, to ensure that the first term in 3.7, coming from the Lelong number of the L^2-metric is non-negative. Let E_i be the reduced components of $\mathcal{X}'_0$ and denote by I the set of all indices i such that E_i is not $p-$exceptional for the log resolution p. For any such i we have $\mathcal{L}'^n \cdot E_i > 0$ and hence if $q = 0$ it follows that $c_i = 0$ and hence $m_i = 0$ for any $i \in I$. But since $\mathcal{X}$ is normal we may, by Hironaka's theorem, take p to be an isomorphism on $p^{-1}(\mathcal{X} - \mathcal{Z})$, where $\mathcal{Z}$ is a subvariety of codimension at least two (containing the singular locus of $\mathcal{X}$) with $\mathcal{X}_0$ reduced at any point in $\mathcal{X} - \mathcal{Z}$ (using $m_i = 0$ for any $i \in I$). Moreover, since $c_i = 0$ for any $i \in I$ we have that $\mathcal{L}$ is isomorphic to $K_{\mathcal{X}/\mathbb{C}}$ on $\mathcal{X} - \mathcal{Z}$ and hence, since the codimension of $\mathcal{X} - \mathcal{Z}$ is at least two $\mathcal{L}$ is the unique extension of $K_{\mathcal{X}/\mathbb{C}}$ from the regular locus of $\mathcal{X}$, which, by definition, means that $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein. The second point follows from Cor 3.3 (together with Lemma 2.1) and the last point from Prop 3.1. $\square$

Remark 3.6. It may be worth pointing out that unless ϕ_t, appearing in the previous theorem, is a (weak) geodesic $\mathcal{D}(\phi^t)$ may not be convex in t, since the corresponding generalized Ding metric Φ' may not be positively curved. But this fact does not effect the proof of the previous theorem, which only uses the the generalized Ding metric is a *difference* of positively curved metrics, where the positive (and a priori singular) part comes from the L^2-metric on the direct image bundle $\pi'_*(\mathcal{L}'+K_{\mathcal{X}'/\mathbb{C}})$.

3.2.1. *An alternative proof of Theorem 1.1 using the singularity structure of the generalized Ding metric.* We can now give a proof of Theorem 1.1 that does not use the deep results about special test configurations in [30], which are based on the Minimal Model Program (nor semi-stable reduction). Given an arbitrary test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ Theorem 3.5 gives that for any bounded geodesic ϕ^t ray emanating from any given metric on $\mathcal{L}$ which is associated to a the given test configuration we have

$$-DF(\mathcal{X}, \mathcal{L}) = \lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi^t) + q, \quad q \geq 0$$

Now, if X admits a Kähler-Einstein metric, that we take to be equal to ϕ^0, then it follows from convexity, as before, that $\lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi^t) \geq 0$ and hence by the previous inequality $-DF(\mathcal{X}, \mathcal{L}) \geq 0$. Moreover if $DF(\mathcal{X}, \mathcal{L}) = 0$ then it must be that $\lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi_t) = 0$, which is equivalent to $\mathcal{D}(\phi_t)$ being affine and $q = 0$. In particular, by Theorem 3.5 the central fiber of $\mathcal{X}$ is generically reduced and, as explained in the proof of Cor 2.9, it thus follows that $\mathcal{X}$ is isomorphic to a product test configuration, as desired (recall that we are assuming that $\mathcal{X}$ is normal).

3.3. **Applications to bounds on the Ricci potential and Perelman's $\lambda-$entropy functional.** Let now X be a Fano manifold and denote by $\mathcal{K}(X)$ the space of all Kähler metrics ω in $c_1(X)$ (equivalently, $\omega = dd^c\phi$ for some strictly positively curved metric ϕ on $-K_X$). In this section we will use the normalization $V := c_1(X)^n := \int_X \omega^n$. Recall that the Ricci potential h_ω is the function on X defined by $dd^c h_\omega = \text{Ric }\omega - \omega$ together with the normalization condition $\int e^{h_\omega}\omega^n/V = 1$, which in terms of the previous notation means that $h_{dd^c\phi} =: = h_\phi = -\log(\frac{(dd^c\phi)^n/V}{e^{-\phi}/\int e^{-\phi}})$.

Note in particular that

$$\left\|1-e^{h_\omega}\right\|_{L^1(X,\omega)}=\left\|\frac{1}{V}(dd^c\phi)^n-\frac{e^{-\phi}}{\int e^{-\phi}}\right\|,$$

where the norm in the rhs is the total variation norm on the space of absolutely continuous probability measures on X.

Next, let $(\mathcal{X},\mathcal{L})$ be a test configuration of a polarized manifold (X,L) and define its "$L^\infty-$norm" by

$$\|(\mathcal{X},\mathcal{L})\|_\infty:=\left\|\frac{d\phi^t}{dt}_{|t=0}\right\|_{L^\infty(X)}, \tag{3.8}$$

where ϕ^t is the (weak) geodesic determined by $\mathcal{X}$, emanating from any fixed reference metric $\phi^0\in\mathcal{H}(X,L)$. The point is that if $\|(\mathcal{X},\mathcal{L})\|_\infty\neq 0$ then the *normalized Donaldson-Futaki invariant* $DF(\mathcal{X},\mathcal{L})/\|(\mathcal{X},\mathcal{L})\|_\infty$ is independent of base changes of $(\mathcal{X},\mathcal{L})$, induced by $\tau\to\tau^m$ (which correspond to reparametrizations of ϕ^t, induced by $t\mapsto mt$). We will be relying on the following lemma which is a special case of a very recent result of Hisamoto [24]:

Lemma 3.7. *The number $\|(\mathcal{X},\mathcal{L})\|_\infty$ is well-defined, i.e. it is independent of ϕ_0.*

Now we can prove the following theorem, using a slight variant of the proof of Theorem 1.3.

Theorem 3.8. *Let X be a Fano manifold. Then*

$$\inf_{\omega\in\mathcal{K}(X)}\left\|1-e^{h_\omega}\right\|_{L^1(X,\omega)}\geq\sup_{(\mathcal{X},\mathcal{L})}\frac{DF(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_\infty},$$

where $(\mathcal{X},\mathcal{L})$ ranges over all test configurations $(\mathcal{X},\mathcal{L})$ such that $\|(\mathcal{X},\mathcal{L})\|_\infty\neq 0$. Moreover, if equality holds and the infimum is attained at some ω and the supremum is attained at $(\mathcal{X},\mathcal{L})$ (with $\mathcal{X}$ normal), then $(\mathcal{X},\mathcal{L})$ is isomorphic to a product test configuration. In particular,

$$\inf_{\omega\in\mathcal{K}(X)}\int h_\omega e^{h_\omega}\frac{\omega^n}{V}\geq\frac{1}{2}\sup_{(\mathcal{X},\mathcal{L})}\left(\frac{DF(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_\infty}\right)^2$$

where the sup ranges over all destabilizing $(\mathcal{X},\mathcal{L})$ (i.e. $DF(\mathcal{X},\mathcal{L})>0$) with the same same necessary conditions for equality as before. In particular, if X is K-unstable then both infimums above are strictly positive.

Proof. Fix $(\mathcal{X},\mathcal{L})$ and $\phi^0\in\mathcal{H}(X,-K_X)$ and denote by ϕ^t the corresponding (weak) geodesic. By convexity of the Ding functional, combined with Theorem 1.3 (using that $q\geq 0$), we have

$$\int_X\left(\frac{1}{V}(dd^c\phi_0)^n-\frac{e^{-\phi}}{\int e^{-\phi}}\right)\frac{d\phi^t}{dt}=-\frac{d}{dt}\mathcal{D}(\phi^t)_{t=0}\geq-\lim_{t\to\infty}\frac{d}{dt}\mathcal{D}(\phi^t)\geq DF(\mathcal{X},\mathcal{L}). \tag{3.9}$$

Applying Hölder's inequality with exponents $(q,p)=(1,\infty)$ thus gives

$$\left\|1-e^{h_\omega}\right\|_{L^1(X,\omega)}\left\|\frac{d\phi^t}{dt}_{|t=0}\right\|_{L^\infty(X)}\geq DF(\mathcal{X},\mathcal{L}) \tag{3.10}$$

and using the independence in the previous lemma then concludes the proof of the first inequality of the Theorem. The second inequality then follows immediately from the classical Csiszar-Kullback-Pinsker inequality between the relative entropy

and the total variation norm. As for the equality case it follows, just as in the second proof of Theorem 1.1, from the equality cases in 3.9. Finally, if X is $K-$unstable then there exists, by definition, a test configuration such that $DF(\mathcal{X},\mathcal{L})>0$ and for any such test configuration the inequality 3.10 forces $\|(\mathcal{X},\mathcal{L})\|_{\infty}>0$, which concludes the proof. $\square$

Recall that in the definition of a test configuration $(\mathcal{X},\mathcal{L})$ we have fixed an action ρ on $\mathcal{L}$ and thus the norm $\|(\mathcal{X},\mathcal{L})\|_{\infty}$ certainly depends on ρ. Indeed, twisting ρ with a character of $\mathbb{C}^*$ shifts the tangent of ϕ^t with a constant. On the other hand, $DF(\mathcal{X},\mathcal{L})$ is independent of such a twist and hence the previous theorem still holds if we replace $\|(\mathcal{X},\mathcal{L})\|_{\infty}$ with its (smaller) normalized version obtained by replacing the $L^{\infty}(X)-$norm in the definition 3.8 with the quotient norm on the quotient space $L^{\infty}(X)/\mathbb{R}$.

Remark 3.9. As pointed out above Lemma 3.7 is a special case of a general result of Hisamoto [24], saying that the measure $(\frac{d\phi^t}{dt})_*MA(\phi^t)$ on $\mathbb{R}$ only depends on the test configuration $(\mathcal{X},\mathcal{L})$ and moreover is equal to the limiting normalized weight measures for the $\mathbb{C}^*-$action, as conjectured by Witt-Nyström [36], who settled the case of product test configurations. In particular, by [24] all the L^p-norms $\|(\mathcal{X},\mathcal{L})\|_p$ of $\frac{d\phi^t}{dt}$ (integrating against $MA(\phi^t)$) only depend on $(\mathcal{X},\mathcal{L})$ and coincide with the limits of the corresponding l^p-norms of the weights $\{\lambda_i^{(k)}\}$. In particular, letting $p\to\infty$ gives Lemma 3.7. Using this the proof of the previous theorem shows that the theorem holds, more generally, when $\|(\mathcal{X},\mathcal{L})\|_{\infty}$ is replaced by $\|(\mathcal{X},\mathcal{L})\|_p$ for $p\in[1,\infty]$ and the L^1-norm with the corresponding L^q-norm, where q is the Young (Hölder) dual of p. In fact, as shown in [5] a similar argument can be used to give a new proof and extend to general L^p-norms Donaldson's lower bound on the Calabi functional [16].

Next, we recall that Perelman's W-functional [40, 53, 54, 21], when restricted to the space $\mathcal{K}(X)$ of all Kähler metrics in $c_1(X)$, is given by

$$W(\omega,f):=\int_X(R+|\nabla f|^2+f)e^{-f}\omega^n$$

(as usually in the Kähler setting where the volume of the metrics is fixed we have set Perelman's parameter τ to be equal to $1/2$). Then Perelman's $\lambda-$entropy functional on $\mathcal{K}(\omega)$ is defined as

$$\lambda(\omega)=\inf_{f\in\mathcal{C}^{\infty}(X):\int e^{-f}\omega^n=1}W(\omega,f)$$

[40, 53, 54, 21] and in particular $\lambda(\omega)\leq W(\omega,0)=nV$.

Corollary 3.10. *Let X be an $n-$dimension Fano manifold. Then*

$$\sup_{\omega\in\mathcal{K}(X)}\lambda(\omega)\leq nV-\frac{1}{2}\sup_{(\mathcal{X},\mathcal{L})}\left(\frac{DF(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_{\infty}}\right)^2$$

where $V=c_1(X)^n$ and $(\mathcal{X},\mathcal{L})$ ranges of all destabilizing test configurations for $(X,-K_X)$. In particular, if X is K-unstable then $\lambda\leq nV-\epsilon$ for some positive number ϵ.

Proof. As explained in [21] $\lambda(\omega)+\int h_\omega e^{h_\omega}\omega^n \leq nV$ (using $W(\omega,f)\leq W(\omega,-h_\omega)$ and one integration by parts) and hence the corollary follows immediately from the previous theorem. □

Remark 3.11. The previous inequality was inspired by the result in [54] and its extension to general non-invariant Kähler metrics in [21], saying that

$$\sup_{\omega\in\mathcal{K}(X)}\lambda(\omega)\leq nV-\sup_{\xi\in\mathrm{Lie}G}H(\xi),$$

with equality if X admits a Kähler-Ricci soliton, where Lie$G$ is the Lie algebra of a maximal compact subgroup in $\mathrm{Aut}_0(X)$and H is a certain concave functional on Lie$G$, defined in [54]. The proof in [21] was based on the convexity of the functional v_{ϕ^t}, while we here use the convexity of the whole Ding functional.

4. The logarithmic setting

Let us briefly recall the more general setting of Kähler-Einstein metrics on log Fano varieties [4] and log K-stability [18, 29, 38]. In a nutshell, this setting is obtained from the previous one by replacing the canonical line bundle K_X with the *log canonical line bundle* $K_{(X,D)}:=K_X+D$ of a given *log pair* (X,D), i.e. X is a normal variety and D is a $\mathbb{Q}-$divisor on X such that K_X+D is defined as a $\mathbb{Q}-$line bundle. For example, (X,D) is said to be a *(weak) log Fano variety* if $-K_{(X,D)}$ is ample (nef and big). A *log Kähler-Einstein metric* ω associated to (X,D) is, by definition, a current ω in $c_1(-K_{(X.D})$, defining a Kähler metric on $X_{reg}-D$, with locally bounded potentials on X and such that

$$\mathrm{Ric}\ \omega-[D]=\omega,$$

holds in the sense of currents, where $[D]$ denotes the current of integration defined by D. Equivalently [4], this means that ω is the curvature current of a locally bounded metric ϕ_{KE} on $-K_{(X,D)}$ satisfying

$$(dd^c\phi_{KE})^n=Ce^{-(\phi_{KE}+\log|s_D|^2)}$$

(for some constant C) in the sense of pluripotential theory, where we recall that $e^{-(\phi+\log|s_D|^2)}$ denotes the measure associated to a metric ϕ on $-K_{(X,D)}$; see section 2.1.1.

The definitions are compatible with log resolutions. More precisely, if $p:\ X'\to X$ is a log resolution of the log pair (X,D), i.e. p is a proper birational morphism such that Suppp^*D+E has simple normal crossings, where E is the exceptional divisor of p, then

$$p^*K_{(X,D)}=K_{X'}+D', \tag{4.1}$$

for a $\mathbb{Q}-$divisor D' on X' (by Hironaka's theorem we may and will assume that p is an isomorphism away from $p^{-1}(X_{sing}\cup\mathrm{Supp}D_{sing})$.

Hence if (X,D) is a weak log Fano variety, then so is (X',D') and ϕ_{KE} is a log Kähler-Einstein metric for (X,D) iff $p^*\phi_{KE}$ is a log Kähler-Einstein metric for (X',D'). In general, a log pair (X,D) is said to have *klt singularities* if the coefficients of D' in formula 4.1 are <1 for any log resolution, which equivalently means that the measure $e^{-(\phi+\log|s_D|^2)}$ has finite mass on X for some (and hence any) locally bounded metric ϕ on $-K_{(X,D)}$.

Example 4.1. If X is smooth and $D = (1-\beta)D_0$ for $\beta \in]0,1]$ and D_0 a smooth hypersurface such that $-(K_X + D)$ is ample, then (X, D) is a log Fano variety. As shown in [22], under the assumption that the log Mabuchi functional (or the log Ding functional) be proper (X, D) admits a log Kähler-Einstein metric ω_{KE}, which moreover has edge-cone singularities along the hypersurface D_0, forming cones of angle $2\pi\beta$ in the transversal directions of D_0 (as shown in [22] the metric even admits a complete polyhomogenous expansion along D_0). One would expect that the existence of a log Kähler-Einstein metric for $(X, (1-\beta)D_0)$ implies that the log Mabuchi functional and the log Ding functional are proper (modulo the existence of holomorphic vector fields tangent to D_0), but this is only known in the usual case when $\beta = 1$. As announced in [22] the results also extend to the general log smooth case, i.e. the case when X is smooth and D is klt with simple normal crossings: $D = \sum(1-\beta_i)D_i$ with $\beta_i \in]0,1]$. However, one of the main points of the approach in the present paper is that it only relies on very weak regularity properties of the metric (the local boundedness of ϕ_{KE}) and that it is independent of any properness assumption. It should also be pointed that, under the assumption that $\beta_i \in]0, 1/2[$, in the previous log smooth case, it is shown in [10] that *any* log Kähler-Einstein metrics has edge-cone singularities, even though the properness of the corresponding functionals is not known.

The notion of K-stability has also been generalized to the log setting (see [18, 29, 38]). Briefly, a test configuration for a log Fano variety (X, D) consists of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) where $L = -K_{(X,D)}$. The $\mathbb{C}^*-$action, applied to the support of D in $\mathcal{X}_1$, induces a $\mathbb{C}^*-$invariant divisor $\mathcal{D}^*$ in $\mathcal{X}^*$ and we denote by $\mathcal{D}$ its closure in $\mathcal{X}$. The corresponding *log Donaldson-Futaki invariant* $DF(\mathcal{X}, \mathcal{L}; \mathcal{D})$ was defined in [29] (by imposing linearity it is enough to consider the case when D is reduced and irreducible). A direct calculation reveals that, up to normalization, the definition in [29] is equivalent to replacing the relative canonical divisor K in the intersection theoretic formula 2.6 with the relative log canonical divisor $K + \mathcal{D}$, defined as a Weil divisor (compare [38]):

$$DF(\mathcal{X}, \mathcal{L}; \mathcal{D}) = -\mu\bar{\mathcal{L}} \cdot \bar{\mathcal{L}} \cdots \bar{\mathcal{L}} - (n+1)(K + \mathcal{D}) \cdot \cdot \bar{\mathcal{L}} \cdots \bar{\mathcal{L}},, \tag{4.2}$$

where now $\mu = n(-(K_X + D)) \cdot L^{n-1}/L^n$. We can hence take the latter formula as the definition of the invariant $DF(\mathcal{X}, \mathcal{L}; \mathcal{D})$. Finally, (X, D) is said to be *log K-polystable* if, for any test configuration, $DF(\mathcal{X}, \mathcal{L}; \mathcal{D}) \leq 0$ with equality iff the test configuration is equivariantly isomorphic to a product test configuration.

Theorem 4.2. *Let (X, D) be a log Fano variety admitting a log Kähler-Einstein metric, where D is an effective $\mathbb{Q}-$divisor on X. Then (X, D) is log K-polystable.*

The theorem thus confirms one direction of the log version of the Yau-Tian-Donaldson conjecture formulated in [29]. The proof of the theorem proceeds, mutatis mutandis, as the proof in the previous case when $D = 0$, and we will hence only give some brief comments on the modifications needed. The key point is that the convexity results for $v_\phi(\tau) := -\log \int_{\mathcal{X}_\tau} e^{-(\phi + \log |s_D|^2)}$, for ϕ a locally bounded metric with positive curvature current on $-K_{(X,D}$, are still valid in the logarithmic setting as long as D is effective (as shown in [4]). Also, as pointed out in the end of [30] the proof of Theorem 2.2 also applies in the log setting and hence it is enough to consider special test configurations in the log setting, where the role of η is now played by the top Deligne pairing of $-(K + \mathcal{D})$. Anyway, the alternative proof in

section 3.2.1 immediately extends to the log setting if one performs a log resolution of $\mathcal{D} + \mathcal{X}_0$.

5. Outlook on the existence problem for Kähler-Einstein metrics on $\mathbb{Q}-$Fano varieties

An immediate consequence of Theorem 3.5 applied to a special test configuration is the following

Corollary 5.1. *Let X be a Fano variety with klt singularities and $\mathcal{X}$ a special test configuration for X such that $DF(\mathcal{X}) < 0$. Fix a smooth and positively curved metric ϕ on $-K_{\mathcal{X}/\mathbb{C}}$ (more generally, local boundedness is enough) and set $\phi^t := \rho^*\phi_\tau$. Then the Ding functional $\mathcal{D}$ and the Mabuchi functional $\mathcal{M}$ both tend to infinity along ϕ^t, as $t \to \infty$.*

Proof. By Theorem 3.5 we have that $\lim_{t\to\infty} \frac{d}{dt}\mathcal{D}(\phi^t) > 0$ and hence $\mathcal{D}(\phi^t) \to \infty$. Since, by an inequality of Bando-Mabuchi, we have $\mathcal{D}(\phi^t) \leq \mathcal{M}(\phi^t)$ (see [4] for the general singular case) this concludes the proof. $\square$

We recall that the Mabuchi functional $\mathcal{M}$ admits a natural extension to the space $\mathcal{H}_b(-K_X)$ taking values in $]-\infty, \infty]$ such that $\mathcal{M}(\phi)$ is finite precisely when the measure $MA(\phi)$ has finite pluricomplex energy and relative entropy [4]. In particular, by the regularity results in [45], $\mathcal{M}(\phi^t)$ is finite, for any fixed t, if the initial metric ϕ_0 is smooth and hence under the assumption in the previous corollary $\mathcal{M}(\phi^t) \to \infty$ tends to infinity as $t \to \infty$. See [43, 12] for related results in the case when the total space $\mathcal{X}$ is assumed smooth.

As will be next briefly explained the previous corollary fits naturally into Tian's program for proving that any K-polystable Fano manifold admits a Kähler-Einstein metric (see the outline in [52]and references therein). It should be pointed out that there has recently has been great progress on Tian's program [19] and we refer the reader to [52, 19] for further background and references. After recalling Tian's program in the smooth setting we will then comment on further complications arising when considering general $\mathbb{Q}$- Fano varieties.

5.1. The case of a smooth Fano variety X.

The starting point of Tian's program is the continuity equation

$$\text{Ric } \omega_t = t\omega_t + (1-t)\eta, \tag{5.1}$$

where ω_0 is a given Kähler metric of positive Ricci curvature η and $t \in [0,1]$ is a fixed parameter. Let I be the set of all t such that a solution ω_t exists. As shown by Aubin $I \cap [0,1[$ is open and non-empty and hence to prove the existence of a Kähler-Einstein metrics, i.e. that $1 \in I$, it is enough to show that I is closed. More precisely, denoting by T the boundary of I and taking $t_i \to T$ we can write $\omega_{t_i} = dd^c\phi_{t_i}$ for suitably normalized metrics ϕ_{t_i} on $-K_X$ (e.g. satisfying $\sup_X(\phi_{t_i} - \phi_0) = 0$) and to show that I is closed it is enough to establish the following C^0-estimate:

$$\sup_X |\phi_{t_i} - \phi_0| \leq C \tag{5.2}$$

(then the higher order estimates follow using the Aubin-Yau C^2-estimate, Evans-Krylov theory and elliptic boot strapping). Before continuing we recall that the following properties hold along the continuity path 5.1 (for a fixed $t_0 \in I$) :

$$(i)\,\text{Ric } \omega_t \geq t_0\omega_t, \quad (ii)\ \ \mathcal{M}(\phi_t) \leq C_0, \tag{5.3}$$

where the first property follows immediately from the fact that $\eta \geq 0$ and the second one from the well-known fact that $\mathcal{M}(\phi_t)$ is decreasing in t.

In order to relate the desired C^0-estimate 5.2 to algebraic properties of X Tian proposed the following conjecture stated in terms of the Bergman function $\rho_\omega^{(k)}(x)$, at level k, associated to a Kähler metric ω on X :

$$\rho_\omega^{(k)}(x) = \sum_{i=0}^{N_k} |s_i^{(\phi)}|^2 e^{-k\phi},$$

where ϕ is any metric on $-K_X$ with curvature form ω and $\{s_i^{(\phi)}\}$ is any base in $H^0(X, -kK_X)$ which is orthonormal wrt the L^2-norm $\|\cdot\|_{k\phi}$ on $H^0(X, -kK_X)$ determined by ϕ, i.e. $\|s\|_{k\phi}^2 = \int_X |s|^2 e^{-k\phi}\omega^n$.

Conjecture 5.2. *(Tian's partial C^0-estimate). Given $t_0 \in]0,1]$, let $\mathcal{K}(X, t_0)$ be the space of all Kähler metrics ω in $c_1(X)$ such that $Ric\omega \geq t_0\omega$. Then there exists a $k > 0$ and $\delta > 0$ such that kL is very ample and for any $\omega \in \mathcal{K}(X, t_0)$,*

$$\inf_X \rho_\omega^{(k)}(x) \geq \delta$$

(more precisely, the conjecture says that k can be chosen arbitrarily large). If the previous conjecture holds then, as follows immediately from the definition of $\rho_\omega^{(k)}$, the desired C^0-estimate holds 5.2 iff

$$\sup_X \left|\phi_{t_i}^{(k)} - \phi_0\right| \leq C \tag{5.4}$$

where now $\phi_{t_i}^{(k)}$ is the Bergman metric at level k determined by ϕ_{t_j}, i.e. $\phi_{t_j}^{(k)} = \frac{1}{k}\log\sum_{i=0}^{N_k} |s_i^{(\phi_{t_j})}|^2$. In other words: $\phi_{t_i}^{(k)}$ is the scaled pull-back of the Fubini-Study metric ϕ_{FS} on $\mathcal{O}(1) \to \mathbb{P}^{N_k}$ under the Kodaira map F_j determined by ϕ_{t_j} :

$$F_j : X \to \mathbb{P}^{N_k}, \quad \phi_{t_i}^{(k)} = F_j^*\phi_{FS}, \quad F_j(X) := V_j$$

i.e. $F_i(x) = [s_0(x) : s_1(x) : \cdots s_{N_k}(x)]$, where now (s_i) is a fixed base, which is orthonormal wrt the L^2-norm determined by ϕ_{t_j} (strictly speaking, due to the choice of base V_i is only determined modulo action of the the unitary group $U(N_k + 1)$, but since this group is compact this fact will be immaterial in the following). After passing to a subsequence we may assume that the projective subvariety $V_j := F_j(X) \subset \mathbb{P}^{N_k}$, converges, in the sense of cycles, to an algebraic cycle V_∞ in $\mathbb{P}^{N_k}$. As indicated by Tian [52] the validity of the previous conjecture would imply that the cycle V_∞ is reduced, irreducible and even defining a *normal* variety and we will thus assume that this is the case (see [19] for a proof under the extra assumption of an upper bound on the Ricci curvature). Next, following Tian [52] we note that, by a compactness argument, there is a one parameter subgroup $\rho : \mathbb{C}^* \to GL(N_k + 1, \mathbb{C})$ such that

$$\sup_X \left|\phi_{t_j}^{(k)} - \rho(\tau_i)^*\phi_{FS}\right| \leq C$$

where $\rho(\tau_i)V_0$ also converges to the normal variety V_∞, as points in the corresponding Hilbert scheme. But then it follows from the universal property of the Hilbert scheme that ρ determines a (special) test configuration $(\mathcal{X}, \mathcal{L})$ with central fiber V_∞ and such that $\rho(\tau)^*\phi_{FS}$ is of the same form as in Cor 5.1 (as formulated in Cor 1.4 in the introduction of the paper).

Now, assuming that X is K-stable (for simplicity we consider the case when X admits no non-trivial holomorphic vector fields, but the general argument is similar)

we have that $DF(\mathcal{X},\mathcal{L})\leq 0$ with equality iff $(\mathcal{X},\mathcal{L})$ is equivariantly isomorphic to a product test-configuration (recall that the total space $\mathcal{X}$ here is automatically normal and even $\mathbb{Q}-$Gorenstein, by Lemma 2.1). In the latter case, $\rho(\tau)^*\phi_{FS}-\phi$ is trivially bounded and hence the desired C^0-estimate 5.2 then holds, showing that X indeed admits a Kähler-Einstein metric (assuming the validity of Tian's conjecture). The main issue is thus to exclude the case of $DF(\mathcal{X},\mathcal{L})<0$ and this is where Cor 5.1 enters into the picture. However, to apply the latter corollary we still need to know that V_∞ has klt singularities. By an observation in [4] this would follow from the normality and $\mathbb{Q}-$Gorenstein property of V_∞ if the regular locus $(V_\infty)_{reg}$ admits a current ω_∞ such that, for some $\epsilon>0$,

$$\text{Ric }\omega_\infty\geq\epsilon\omega_\infty\ \text{ on }(V_\infty)_{reg}\tag{5.5}$$

More precisely, in order to make sense of the previous condition we also assume that ω_∞ has locally bounded potentials ϕ_∞ on $(V_\infty)_{reg}$ and that the corresponding Monge-Ampère measure ω_∞^n has a local density of the form $e^{-\psi_\infty}$ with ψ_∞ in L^1_{loc} so that Ric $\omega_\infty:=dd^c\psi_\infty$.

Lemma 5.3. *Let Y be a normal variety which admits a current ω_∞ on its regular locus with strictly positive Ricci curvature in the sense of 5.5. Then Y has klt singularities.*

Thus, assuming the validity of Tian's Conjecture 5.2 and the existence of a current ω_∞ as in 5.5 we deduce from Cor 5.1 that if the second alternative $DF(\mathcal{X},\mathcal{L})<0$ holds, then the Ding functional $\mathcal{D}$ tends to infinity along $\rho(\tau_i)^*\phi_{FS}$ and hence it is unbounded from above along $\phi_{t_j}^{(k)}$. But this implies that $\mathcal{D}$ is also unbounded along the original sequence ϕ_{t_j} (also using that if $|\psi-\psi'|\leq C$ then $|\mathcal{D}(\psi)-\mathcal{D}(\psi')|\leq 2C$, as follows immediately from the definition 2.8). But, since $\mathcal{D}\leq\mathcal{M}$, this contradicts the property (ii) in formula 5.3 hence it must be that the first alternative, $DF(\mathcal{X},\mathcal{L})=0$, holds and thus X admits a Kähler-Einstein metric, as desired.

5.1.1. *Comments on the existence of a current ω_∞ as in 5.5.* It seems natural to expect that (given the validity of Tian's Conjecture 5.2) the existence of ω_∞ is automatic and that ω_∞ may be obtained as a suitable limit of the sequence ω_{t_j}. For example, this is the case for a general toric (possibly K-unstable) toric variety, as follows from a result of Li [28]. Moreover, the convergence theory of Cheeger-Colding-Tian [11] suggest that any Gromov-Hausdorff limit of (X,ω_{t_j}) (which exists by Gromov compactness) has conical singularities and induces a Kähler current ω_∞ on V_∞, which has conical singularities on the regular locus of V_∞. In particular, this would mean that it satisfies the regularity assumptions underlying the condition 5.5. In fact, the idea of using Gromov-Haussdorf convergence to establish Conjecture 5.2 has been emphasized by Tian [52] and was recently successfully used by Donaldson-Sun in [19] to show that the conjecture 5.5 holds under the further assumption of an *upper bound* on the Ricci curvature. Moreover, under the latter assumptions the results in [19] also give a limiting Kähler metric ω_∞ satisfying the condition 5.5. The point is that the upper bound on the Ricci curvature, ensures, according to the results in [11], that the singularity locus of the Gromov-Haussdorf limit is of real codimension strictly bigger than two (more precisely, of codimension at least four). However, in general one expects a metric singular locus of real codimension two coming from conical singularities of the metric (compare [11]).

5.2. **Towards the case of $\mathbb{Q}-$Fano varieties.** Let us finally discuss the complications that arise when trying to generalize Tian's program to the case of singular K-polystable Fano varieties X (by [37] such a Fano variety X automatically has klt singularities). Taking η to be a smooth semi-positive form in $c_1(X)$ the continuity equations 5.1 are defined as before and, by the results in [4], the set I is still non-empty, i.e. $T > 0$ (using the positivity of the alpha invariant of X). The solutions ω_t define Kähler forms on X_{reg} with volume $c_1(X)^n/n!$. Next, we note that Tian's conjecture admits a natural generalization to general $\mathbb{Q}-$Fano varieties if one uses the notion of (singular) Ricci curvature appearing in [4] (and similarly for general log Fano varieties (X, D)). However, one new difficulty that arises is the *openness* of I. From the point of view of the implicit function theorem the problem is to find appropriate Banach spaces, encoding the singularities of X (the uniqueness of solutions to the formally linearized version of equation 5.1, for $t \in]0, 1[$, follows from the results in [4]). On the other hand, another approach could be to use the following lemma, where the properness refers to the exhaustion function defined by the $J-$functional (see [4] for the singular case).

Lemma 5.4. *The set I is open iff the twisted Ding (Mabuchi) functional $\mathcal{D}_t$ (associated to the twisting form $(1-t)\eta$) is proper for any $t \in I$.*

Proof. If $\mathcal{D}_t$ is proper, then it follows from the results in [4] that a solution ω_t exists. Conversely, if a solution ω_t exists and I is open, i.e. solutions $\omega_{t+\delta}$ exist for δ sufficiently small, then it follows from the convexity of $\mathcal{D}_{t+\delta}$ along weak geodesics that $\mathcal{D}_{t+\delta} \geq C$. But since δ may be taken to be positive this implies that $\mathcal{D}_t$ is proper (and even coercive; compare [4]). □

Note that in the case $n = 2$ it is a basic fact that a projective variety X has klt singularities iff it has quotient singularities (defining an orbifold structure on X) and hence the two-dimensional Fano varieties are precisely the orbifold Del Pezzo surfaces. In the general Fano orbifold case, if one takes η to be an orbifold Kähler metric, the usual implicit function theorem applies to give that I above is indeed open. For the case of K-polystable Del Pezzo surfaces with canonical singularities (i.e. ADE singularities) the existence of Kähler-Einstein metrics was established very recently in [39], using a different method, thus generalizing the case of smooth Del Pezzo surfaces settled by Tian [51].

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